Periodic oscillation for a coupled FHN model with delays

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Abstract: In this paper, a coupled FitzHugh-Nagumo (FHN) model with five discrete delays is studied. The oscillatory behavior of the solutions is investigated. We extend the result in the literature from one or two coupled FHN models to a four-coupled system. By means of the method of mathematical analysis, two sufficient conditions to guarantee the oscillation of the solutions are provided, and some simulations are given to indicate the correctness of the present criterion, in which the amplification functions arctan (u) and tanh (u) are to be used.

Keywords: four-coupled FHN model, delay, instability, oscillation

I. Introduction

In recent years, the coupled FitzHugh-Nagumo (FHN) model has been a hot research topic. Many researchers have studied various FHN models for their properties, such as stability, synchronization, bifurcation, and chaos. For example, Jia et al. investigated the following delayed FHN model

$$\begin{cases} x_1'(t) = c\left(y_1(t) + x_1(t) - \frac{1}{3}x_1^3(t)\right) + \gamma\big(x_1(t) - x_2(t-\tau)\big), \\ y_1'(t) = -\frac{1}{c}\big(x_1(t) - a + by_1(t)\big), \\ x_2'(t) = c\left(y_2(t) + x_2(t) - \frac{1}{3}x_2^3(t)\right) + \gamma\big(x_2(t) - x_1(t-\tau)\big), \\ y_2'(t) = -\frac{1}{c}\big(x_2(t) - a + by_2(t)\big), \end{cases} \tag{1}$$

where $x_i(t)$, and $y_i(t)$ $(i = 1, 2)$ represent the voltage across the cell membrane, and the recovery state of the resting membrane of a neuron, respectively. a, b, and c are parameters, and γ is the coupling strength between the network elements. By employing the corresponding characteristic equation, the stability of the equilibrium point involved the delay-independence stability and delay dependence stability was analyzed. Sufficient conditions for the existence of a Hopf bifurcation and the direction of Hopf bifurcation were obtained by the normal form theory and the center manifold theorem for functional differential equations [1]. By selecting the coupling strength and time delay as the bifurcation parameters, Wang et al. discussed the following FHN model:

$$\begin{cases} v_1'(t) = -v_1^3(t) + av_1(t) - v_2(t) + c\tanh\big(v_3(t-\tau)\big), \\ v_2'(t) = v_1(t) - b_1v_2(t), \\ v_3'(t) = -v_3^3(t) + av_3(t) - v_4(t) + c\tanh\big(v_1(t-\tau)\big), \\ v_4'(t) = v_3(t) - b_2v_4(t). \end{cases} \tag{2}$$

The authors performed the bifurcation analysis and numerical simulations. Some interesting phenomena, such as the existence of a stable fixed point, a stable periodic solution near the Hopf-pitchfork critical point, were presented [2]. By employing coupling strength and time delay as bifurcation parameters, a coupled non-identical FitzHugh-Nagumo neuron with delayed synaptic connection was considered. The pitchfork bifurcation of a non-trivial rest point was exhibited [3]. Achouri et al. investigated a two-delays FHN system as follows:

$$\begin{cases} x_1'(t) = -x_1(t)(x_1(t)-1)(x_1(t)-a) - y_1(t) + c_1 f_1\big(x_2(t-\tau_2)\big), \\ y_1'(t) = b_1\big(x_1(t) - \gamma_1 y_1(t)\big), \\ x_2'(t) = -x_2(t)(x_2(t)-1)(x_2(t)-a) - y_2(t) + c_2 f_2\big(x_1(t-\tau_1)\big), \\ y_2'(t) = b_2\big(x_2(t) - \gamma_2 y_2(t)\big). \end{cases} \tag{3}$$

By using the normal form theory and reduction on the center manifold, the truncated normal form was obtained for model (3), and throughout the bifurcation diagram, its dynamical behavior was studied. A third and fourth-degree exponential polynomial was considered. The critical values where the Bogdanov–Takens bifurcation occurred were derived [4]. In recent years, researchers have developed many FHN models [5–8]. Yamakou et al. considered a stochastic FHN neuron model in the excitable regime that embeds a leaky integrate-and-fire model [9]. Gambino et al. conducted an analysis on the excitable regime for an FHN system [10]. In [11], the authors provided a phase reduction approach to reduce the dimension of limit-cycle oscillators for an FHN model. Concha and Garrido proposed two methodologies for estimating the parameters of an FHN neuron model [12]. Bisquert made a frequency domain analysis for an FHN model [13]. Ahsan et al. introduced an electronic circuit that mimics the functionality of a biological spiking neuron following the FHN model [14]. Ara discussed the parameter estimation of an FHN model [15]. Some researchers deal with fractional-order coupled FHN neuronal models [16-22]. Different from models (1) - (3), Zhen and Xu considered a three-coupled FHN model:

$$\begin{cases} u_1'(t) = -\frac{1}{3}u_1^3(t) + cu_1^2(t) + du_1(t) - u_2(t) + \alpha u_1^2(t) + \beta(f(u_3(t-\tau) + f(u_5(t-\tau))), \\ u_2'(t) = \varepsilon\big(u_1(t) - bu_2(t)\big), \\ u_3'(t) = -\frac{1}{3}u_3^3(t) + cu_3^2(t) + du_3(t) - u_4(t) + \alpha u_3^2(t) + \beta(f(u_1(t-\tau) + f(u_5(t-\tau))), \\ u_4'(t) = \varepsilon\big(u_3(t) - bu_4(t)\big), \\ u_5'(t) = -\frac{1}{3}u_5^3(t) + cu_5^2(t) + du_5(t) - u_6(t) + \alpha u_5^2(t) + \beta(f(u_1(t-\tau) + f(u_3(t-\tau))), \\ u_6'(t) = \varepsilon\big(u_5(t) - bu_6(t)\big), \end{cases} \tag{4}$$

where α, β measure the synaptic strength of self-connection and neighborhood-interaction, respectively, f is a sufficiently smooth, bounded sigmoid amplification function. The Bautin bifurcation for model (4) has been discussed [23]. In [24], the authors discussed lag synchronization of n-coupled delayed FHN neural networks. Motivated by the above models, in this paper, we shall study the dynamical behavior of the following five-coupled delayed FHN system:

$$\begin{cases} u_1'(t) = -\frac{1}{3}u_1^3(t) + c_1u_1^2(t) + d_1u_1(t) - k_2u_2(t) + \alpha_1u_1^2(t) + \beta_{13}f(u_3(t-\tau_3)) \\ \quad +\beta_{15}f\big(u_5(t-\tau_5)\big) + \beta_{17}f\big(u_7(t-\tau_7)\big) + \beta_{19}f\big(u_9(t-\tau_9)\big), \\ u_2'(t) = \varepsilon_1\big(u_1(t) - b_2u_2(t)\big), \\ u_3'(t) = -\frac{1}{3}u_3^3(t) + c_3u_3^2(t) + d_3u_3(t) - k_4u_4(t) + \alpha_3u_3^2(t) + \beta_{31}f(u_1(t-\tau_1)) \\ \quad +\beta_{35}f\big(u_5(t-\tau_5)\big) + \beta_{37}f\big(u_7(t-\tau_7)\big) + \beta_{39}f\big(u_9(t-\tau_9)\big), \\ u_4'(t) = \varepsilon_3\big(u_3(t) - b_4u_4(t)\big), \\ u_5'(t) = -\frac{1}{3}u_5^3(t) + c_5u_5^2(t) + d_5u_5(t) - k_6u_6(t) + \alpha_5u_5^2(t) + \beta_{51}f(u_1(t-\tau_1)) \\ \quad +\beta_{53}f\big(u_3(t-\tau_3)\big) + \beta_{57}f\big(u_7(t-\tau_7)\big) + \beta_{59}f\big(u_9(t-\tau_9)\big), \\ u_6'(t) = \varepsilon_5\big(u_5(t) - b_6u_6(t)\big), \\ u_7'(t) = -\frac{1}{3}u_7^3(t) + c_7u_7^2(t) + d_7u_7(t) - k_8u_8(t) + \alpha_7u_7^2(t) + \beta_{71}f(u_1(t-\tau_1)) \\ \quad +\beta_{73}f\big(u_3(t-\tau_3)\big) + \beta_{75}f\big(u_5(t-\tau_5)\big) + \beta_{79}f\big(u_9(t-\tau_9)\big), \\ u_8'(t) = \varepsilon_7\big(u_7(t) - b_8u_8(t)\big), \\ u_9'(t) = -\frac{1}{3}u_9^3(t) + c_9u_9^2(t) + d_9u_9(t) - k_{10}u_{10}(t) + \alpha_9u_9^2(t) + \beta_{91}f(u_1(t-\tau_1)) \\ \quad +\beta_{93}f\big(u_3(t-\tau_3)\big) + \beta_{95}f\big(u_5(t-\tau_5)\big) + \beta_{97}f\big(u_7(t-\tau_7)\big), \\ u_{10}'(t) = \varepsilon_9\big(u_9(t) - 10u_{10}(t)\big), \end{cases} \quad , \tag{5}$$

where all parameters are real constants, time delays $\tau_i \geq 0$ $(i = 1,3,5,7,9)$, $f(0) = 0$, and $f(u_1(t-\tau_1))$ are sufficiently smooth sigmoid amplification functions such as tanh $(u_i(t-\tau_i))$, and arctan $(u_i(t-\tau_i))$. Model (5) includes five delays. Our goal is to investigate the periodic oscillation of the solutions for model (5). Obviously, one can use the bifurcation method since bifurcation can lead to periodic oscillations. However, if the five delays are different real numbers, the bifurcating equation is very complex and is not easy to discuss the bifurcating values. In the present paper, we will use the method of mathematical analysis to discuss the existence of periodic oscillation for model (5).

II. Preliminaries

For the system (5), we first prove the following lemma.

Lemma 1. All solutions of the system (5) are bounded.

Proof To prove the boundedness of the solutions for the system (5), consider a Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{10} u_i^2(t)$. Since $f(u_i)$ are bounded function, we have

$$\begin{aligned} V'(t)|_{(5)} &= \sum_{i=1}^{10} u_i(t)u_i'(t) \\ &\leq -\frac{1}{3}\sum_{i=0}^{4} u_{2i+1}^4(t) + \sum_{i=0}^{4} c_{2i+1}u_{2i+1}^3(t) + \sum_{i=1}^{5} Mu_{2i}^2(t) \\ &\quad + \sum_{i=1}^{5} Nu_{2i-1}(t)u_{2i}(t) \end{aligned} \tag{6}$$

where M, N are some positive constants. Note that when $u_i(t) \to \infty$, $u_i^4(t)$ are higher order infinity than $u_i^3(t), u_i^2(t)$, $u_{2i-1}(t)u_{2i}(t)$. Since $-\frac{1}{3} < 0$, so, there exists a suitably large number $L > 0$ such that $V'(t)|_{(5)} < 0$ as $u_i > L$ $(i = 1,2,\cdots,10)$. This means that all solutions of the system (5) are bounded.

Now suppose that $[u_1^*, u_2^*, \cdots, u_{10}^*]^T$ is an equilibrium point of the system (5), then the Taylor expansion for functions $f(u_i(t-\tau_i))$ at the equilibrium point u_i^* $(i = 1,3,5,7,9)$.

$$f\big(u_i(t-\tau_i)\big) = f(u_i^*) + \sum_{i=1}^{\infty} \frac{f^n(u_i^*)}{n!}(u_i(t-\tau_i) - u_i^*)^n \quad (i = 1,3,5,7,9) \tag{7}$$

For the sake of simplicity, make the change of the variables $u_i(t) \to u_i(t) - u_i^*$ $(i = 1, 2, \cdots, 10)$. Noting that

$$\begin{cases} -\frac{1}{3}(u_1^*)^3 + c_1(u_1^*)^2 + d_1u_1^* - k_2u_2^* + \alpha_1(u_1^*)^2 + \beta_{13}f(u_3^*) \\ +\beta_{15}f(u_5^*) + \beta_{17}f(u_7^*) + \beta_{19}f(u_9^*) = 0, \\ \varepsilon_1(u_1^* - b_2u_2^*) = 0, \\ -\frac{1}{3}(u_3^*)^3 + c_3(u_3^*)^2 + d_3u_3^* - k_4u_4^* + \alpha_3(u_3^*)^2 + \beta_{31}f(u_1^*) \\ +\beta_{35}f(u_5^*) + \beta_{37}f(u_7^*) + \beta_{39}f(u_9^*) = 0, \\ \varepsilon_3(u_3^* - b_4u_4^*) = 0, \\ -\frac{1}{3}(u_5^*)^3 + c_5(u_5^*)^2 + d_5u_5^* - k_6u_6^* + \alpha_5(u_5^*)^2 + \beta_{51}f(u_1^*) \\ +\beta_{53}f(u_3^*) + \beta_{57}f(u_7^*) + \beta_{59}f(u_9^*) = 0, \\ \varepsilon_5(u_5^* - b_6u_6^*) = 0, \\ -\frac{1}{3}(u_7^*)^3 + c_7(u_7^*)^2 + d_7u_7^* - k_8u_8^* + \alpha_7(u_7^*)^2 + \beta_{71}f(u_1^*) \\ +\beta_{73}f(u_3^*) + \beta_{75}f(u_5^*) + \beta_{79}f(u_9^*) = 0, \\ \varepsilon_7(u_7^* - b_8u_8^*) = 0, \\ -\frac{1}{3}(u_9^*)^3 + c_9(u_9^*)^2 + d_9u_9^* - k_{10}u_{10}^* + \alpha_9(u_9^*)^2 + \beta_{91}f(u_1^*) \\ +\beta_{93}f(u_3^*) + \beta_{95}f(u_5^*) + \beta_{97}f(u_7^*) = 0, \\ \varepsilon_9(u_9^* - b_{10}u_{10}^*) = 0, \end{cases}, \tag{8}$$

We have the following system:

$$\begin{cases} u_1'(t) = -\frac{1}{3}u_1^3(t) + (c_1 - u_1^* + \alpha_1)u_1^2(t) + d_{11}u_1(t) - k_2u_2(t) + \alpha_1u_1^2(t) \\ +\beta_{13}\sum_{n=1}^{\infty}\frac{f^n(u_3^*)}{n!}u_3^n(t-\tau_3) + \beta_{15}\sum_{n=1}^{\infty}\frac{f^n(u_5^*)}{n!}u_5^n(t-\tau_5) \\ +\beta_{17}\sum_{n=1}^{\infty}\frac{f^n(u_7^*)}{n!}u_7^n(t-\tau_7) + \beta_{19}\sum_{n=1}^{\infty}\frac{f^n(u_9^*)}{n!}u_9^n(t-\tau_9), \\ u_2'(t) = \varepsilon_1\big(u_1(t) - b_2u_2(t)\big), \\ u_3'(t) = -\frac{1}{3}u_3^3(t) + (c_3 - u_3^* + \alpha_3)u_3^2(t) + d_{33}u_3(t) - k_4u_4(t) + \alpha_3u_3^2(t) \\ +\beta_{31}\sum_{n=1}^{\infty}\frac{f^n(u_1^*)}{n!}u_1^n(t-\tau_1) + \beta_{35}\sum_{n=1}^{\infty}\frac{f^n(u_5^*)}{n!}u_5^n(t-\tau_5) \\ +\beta_{37}\sum_{n=1}^{\infty}\frac{f^n(u_7^*)}{n!}u_7^n(t-\tau_7) + \beta_{39}\sum_{n=1}^{\infty}\frac{f^n(u_9^*)}{n!}u_9^n(t-\tau_9), \\ u_4'(t) = \varepsilon_3\big(u_3(t) - b_4u_4(t)\big), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_9'(t) = -\frac{1}{3}u_9^3(t) + (c_9 - u_9^* + \alpha_9)u_9^2(t) + d_{99}u_9(t) - k_{10}u_{10}(t) + \alpha_9u_9^2(t) \\ +\beta_{91}\sum_{n=1}^{\infty}\frac{f^n(u_1^*)}{n!}u_1^n(t-\tau_1) + \beta_{93}\sum_{n=1}^{\infty}\frac{f^n(u_3^*)}{n!}u_3^n(t-\tau_3) \\ +\beta_{95}\sum_{n=1}^{\infty}\frac{f^n(u_5^*)}{n!}u_5^n(t-\tau_5) + \beta_{97}\sum_{n=1}^{\infty}\frac{f^n(u_7^*)}{n!}u_7^n(t-\tau_7), \\ u_{10}'(t) = \varepsilon_9\big(u_9(t) - 10u_{10}(t)\big), \end{cases} \tag{9}$$

where $d_{ii} = d_i - (u_1^*)^2 + 2c_iu_i^* + 2\alpha_iu_i^*$ $(i = 1, 3, 5, 7, 9)$. Obviously, the trivial equilibrium point of system (9) corresponds to the equilibrium point $[u_1^*, u_2^*, \cdots, u_{10}^*]^T$ of the system (5).
The linearized system of (9) is the following:

$$\begin{cases} u_1'(t) = d_{11}u_1(t) - k_2u_2(t) + b_{13}u_3(t-\tau_3) + b_{15}u_5(t-\tau_5) + b_{17}u_7(t-\tau_7) + b_{19}u_9(t-\tau_9), \\ u_2'(t) = \varepsilon_1\big(u_1(t) - b_2u_2(t)\big), \\ u_3'(t) = d_{33}u_3(t) - k_4u_4(t) + b_{31}u_1(t-\tau_1) + b_{35}u_5(t-\tau_5) + b_{37}u_7(t-\tau_7) + b_{39}u_9(t-\tau_9), \\ u_4'(t) = \varepsilon_3\big(u_3(t) - b_4u_4(t)\big), \\ u_5'(t) = d_{55}u_5(t) - k_6u_6(t) + b_{51}u_1(t-\tau_1) + b_{53}u_3(t-\tau_3) + b_{57}u_7(t-\tau_7) + b_{59}u_9(t-\tau_9), \\ u_6'(t) = \varepsilon_5\big(u_5(t) - b_6u_6(t)\big), \\ u_7'(t) = d_{77}u_7(t) - k_8u_8(t) + b_{71}u_1(t-\tau_1) + b_{73}u_3(t-\tau_3) + b_{75}u_5(t-\tau_5) + b_{79}u_9(t-\tau_9), \\ u_8'(t) = \varepsilon_7\big(u_7(t) - b_8u_8(t)\big), \\ u_9'(t) = d_{99}u_9(t) - k_{10}u_{10}(t) + b_{91}u_1(t-\tau_1) + b_{93}u_3(t-\tau_3) + b_{95}u_5(t-\tau_5) + b_{97}u_7(t-\tau_7), \\ u_{10}'(t) = \varepsilon_9\big(u_9(t) - 10u_{10}(t)\big), \end{cases} \tag{10}$$

where $b_{ij} = \beta_{ij}f\big(u_j^*\big)$ $(i,j = 1,3,5,7,9)$. The system (10) can be expressed in the following matrix form:

$$u'(t) = Du(t) + Bu(t-\tau) \tag{11}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_{10}(t)]^T$, $u(t-\tau) = [u_1(t-\tau_1), 0, u_3(t-\tau_3), \cdots, u_9(t-\tau_9), 0]^T$, D and B both are 10×10 matrices.

$$D = (d_{ij})_{10\times10} = \begin{pmatrix} d_{11} & -k_2 & & 0 & 0 & \cdots & 0 & 0 & \\ \varepsilon_1 & -\varepsilon_1b_2 & & 0 & 0 & \cdots & 0 & 0 & \\ 0 & 0 & d_{33} & -k_4 & & \cdots & 0 & 0 & \\ 0 & 0 & \varepsilon_3 & -\varepsilon_3b_4 & & \cdots & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & d_{99} & -k_{10} & & \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_9 & -\varepsilon_9b_{10} & & \end{pmatrix},$$

$$B = (b_{ij})_{10\times10} = \begin{pmatrix} 0 & 0 & b_{13} & 0 & \cdots & b_{19} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{31} & 0 & 0 & 0 & \cdots & b_{39} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{91} & 0 & b_{93} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then we have the following lemma:

Lemma 2. If matrix $A = D + B$ is a nonsingular matrix for selected parameters, then there exists a unique equilibrium point of the system (5).

Proof Since the trivial equilibrium of the system (9) corresponds to the equilibrium point $[u_1^*, u_2^*, \cdots, u_{10}^*]^T$ of the system (5), we only need to prove that system (11) then system (9) has a unique trivial equilibrium point. Assume that $\boldsymbol{v}^*$ is an equilibrium point of the system (11), then we have

$$D\boldsymbol{v}^* + B\boldsymbol{v}^* = A\boldsymbol{v}^* = \mathbf{0} \tag{12}$$

According to Cramer's Rule of linear algebraic theory, the system (12) has a unique trivial solution since A is a nonsingular matrix, implying that the system (10) has a unique trivial equilibrium point. The proof is completed.

In the following, we provide two theorems to guarantee the existence of periodic solutions for model (5).

III. The existence of periodic oscillatory solutions

Theorem 1 Assume that the system (9) has a unique trivial equilibrium point. Let $\rho_1, \rho_2, \cdots, \rho_{10}$, $\delta_1, \delta_2, \cdots, \delta_{10}$ are characteristic values of matrix D and B, respectively. Suppose that all ρ_i have negative real parts, and $Re(\rho_i + \delta_i) < 0$ $(i = 1, 2, \cdots, 10)$, then the trivial solution of the system (10) is stable, implying that the trivial solution of the system (5) is stable. If there is some ρ_i, such that $Re(\rho_i) > 0$, or there exists some δ_i, say δ_1 such that $Re(\delta_1) > |Re(\rho_1)|$, then the unique equilibrium point of system (9) is unstable, implying that there exists a periodic oscillatory solution in the system (5).

Proof According to the property of the delayed differential equation, the condition $Re(\rho_i + \delta_i) < 0$ $(i = 1, 2, \cdots, 10)$ ensures the stability of the trivial equilibrium point of the system (9). Noting that $f(u_i)$ are sufficiently smooth sigmoid amplification functions, and the nonlinear terms of the system (5) are higher-order infinitesimals as $u_i \to 0$. The stability of the trivial equilibrium of the system (9) guarantees the stability of the equilibrium point of the system (5). To discuss the instability of the equilibrium point of the system (5), we only need to deal with the instability of the trivial solution of the system (9). First we discuss the linearized system (11). Since $\rho_1, \rho_2, \cdots, \rho_{10}$, $\delta_1, \delta_2, \cdots, \delta_{10}$ are characteristic values of matrix D and B, respectively, we have five characteristic values of matrix B are zeros. Then the characteristic equations corresponding to the system (11) are the following:

$$\prod_{i=1}^{5}\left(\lambda - \rho_i - \delta_i e^{-\lambda\tau_i}\right) = 0 \qquad (13)$$

or

$$\prod_{i=6}^{10}(\lambda - \rho_i) = 0 \qquad (14)$$

Thus, we are led to investigate the nature of the roots of the above equations.

If $Re(\rho_6) > 0$, then equation (14) has a positive real part eigenvalue. If equation (13) holds, we have

$$\lambda - \rho_1 - \delta_1 e^{-\lambda\tau_1} = 0 \qquad (15)$$

We show that there exists a positive real part eigenvalue of equation (15) under the assumptions of Theorem 1. Indeed, if $Re(\delta_1) > |Re(\rho_1)|$, setting $\lambda = \lambda_1 + i\lambda_2, \rho_1 = \rho_{11} + i\rho_{12}$, $\delta_1 = \delta_{11} + i\delta_{12}$, where $\lambda_1 = Re(\lambda), \lambda_2 = Im(\lambda)$, $\rho_{11} = Re(\rho_1)$, $\rho_{12} = Im(\rho_1)$, $\delta_{11} = Re(\delta_1), \delta_{12} = Im(\delta_1)$. Separating the real part and imaginary part of the equation (15), we get

$$\lambda_1 = \rho_{11} + \delta_{11}e^{-\lambda_1\tau_1}\cos(\lambda_2\tau_1) - \delta_{12}e^{-\lambda_1\tau_1}\sin(\lambda_2\tau_1) \qquad (16)$$

We show that equation (16) has a positive root. Let

$$g(\lambda_1) = \lambda_1 - \rho_{11} - \delta_{11}e^{-\lambda_1\tau_1}\cos(\lambda_2\tau_1) + \delta_{12}e^{-\lambda_1\tau_1}\sin(\lambda_2\tau_1) \qquad (17)$$

Obviously, $g(\lambda_1)$ is a continuous function of λ_1. Noting that $\delta_{11} > |\rho_{11}|$, then $g(0) = -\rho_{11} - \delta_{11}\cos(\lambda_2\tau_1) + \delta_{12}\sin(\lambda_2\tau_1) \le -(\rho_{11} + \delta_{11}) < 0$ as $\lambda_2\tau_1 \sim 2n\pi$, where n is an integer number. Obviously, there exists a suitably large λ_1, say $\lambda_1^*(> 0)$ such that

$$g(\lambda_1^*) = \lambda_1^* - \rho_{11} - \delta_{11}e^{-\lambda_1^*\tau_1}\cos(\lambda_2\tau_1) + \delta_{12}e^{-\lambda_1^*\tau_1}\sin(\lambda_2\tau_1) > 0. \qquad (18)$$

By the Intermediate Value Theorem, there exists a λ_1, say $\lambda_{10} \in (0,\ \lambda_1^*)$ such that $g(\lambda_{10}) = 0$, implying that there is a positive real part characteristic value of equation (16). This means that the trivial solution of system (11) is unstable. It suggests that the trivial solution of system (9) is also unstable, implying that the unique equilibrium point of the system (5) is unstable. This instability of the unique equilibrium point, together with the boundedness of the solutions, will force system (5) to generate an oscillatory solution [25, 26]. The proof is completed.

Now setting $\mu(D) = \max_{1\le j\le 10}(d_{jj} + \sum_{i=1,i\neq j}^{10}|d_{ij}|)$, $\|B\| = \max\{\sum_{i=1}^{4}|b_{2i+1,1}|, \sum_{i=0,i\neq 1}^{4}|b_{2i+1,3}|, \sum_{i=0,i\neq 2}^{4}|b_{2i+1,5}|, \sum_{i=0,i\neq 3}^{4}|b_{2i+1,7}|, \sum_{i=0,i\neq j}^{3}|b_{2i+1,9}|\}$ [27]. Then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality holds

$$\|B\|e\tau_* > e^{|\mu(D)|\tau_*} \tag{19}$$

where $\tau_* = \min\{\tau_1, \tau_3, \cdots, \tau_9\}$. Then the unique trivial equilibrium point in system (11) is unstable, which suggests that the trivial equilibrium point is unstable in system (9), implying that system (5) generates a periodic oscillatory solution.

Proof To prove the instability of the trivial solution in system (9), we only need to consider system (11). Let $w(t) = \sum_{i=1}^{10}|u_i(t)|$. So $w(t) > 0$ and $w(t)$ satisfying:

$$w(t) \le \mu(D) + \|B\|w(t-\tau_*) \tag{20}$$

Specifically, consider the equation

$$z(t) = \mu(D) + \|B\|z(t-\tau_*) \tag{21}$$

Obviously, $w(t) \le z(t)$. If the trivial solution of equation (21) is unstable, then the trivial solution of (20) is still unstable. The characteristic equation associated with equation (21) is given by

$$\lambda = \mu(D) + \|B\|e^{-\lambda\tau_*} \tag{22}$$

We claim that there exists a positive root of (22) under the condition (19). Assume that the equation (22) has a negative root, say λ^*, then

$$|\lambda^*| \ge \|B\|e^{|\lambda^*|\tau_*} - |\mu(D)| \tag{23}$$

Using inequality $e^x \ge ex$, we have

$$|\lambda^*| + |\mu(D)| \ge \|B\|e^{|\lambda^*|\tau_*} = \frac{\|B\|e^{(|\lambda^*|+|\mu(D)|)\tau_*}}{e^{|\mu(D)|\tau_*}} \ge \frac{\|B\|e\,(|\lambda^*|+|\mu(D)|)\tau_*}{e^{|\mu(D)|\tau_*}} \tag{24}$$

Thus, we have

$$e^{|\mu(D)|\tau_*} \ge \|B\|e\tau_* \tag{25}$$

A contradiction with inequality (19). Therefore, equation (21) has a positive characteristic root, implying that the trivial equilibrium of the system (11) is unstable. According to the property of the delayed differential equation, when $\tau_i \ge \tau_*(i = 1, 3, \cdots, 9)$, the trivial solution of the equation (11) is still unstable, implying that the unique trivial equilibrium point of the system (9) is unstable. Similar to Theorem 1, there exists a periodic oscillatory solution of the system (5). The proof is completed.

IV. Simulation result

This simulation is based on the system (5). Firstly, the amplification function is selected as $\tan^{-1}(u_i)$, the parameters are selected as follows:

$c_1 = 0.42, c_3 = 0.45, c_5 = 0.48, c_7 = 0.50, c_9 = 1.24, d_1 = 0.82, d_3 = 0.85, d_5 = 0.80, d_7 = 0.86, d_9 = 0.88, k_2 = 0.2, k_4 = 0.3, k_6 = 0.4, k_8 = 0.5, k_{10} = 0.55, b_2 = 0.12, b_4 = 0.14, b_6 = 0.18, b_8 = 0.16, b_{10} = 0.20, \varepsilon_1 = 0.84, \varepsilon_3 = 0.85, \varepsilon_5 = 0.88, \varepsilon_7 = 0.82, \varepsilon_9 = 0.83, \alpha_1 = 0.62, \alpha_3 = 0.65, \alpha_5 = 0.60, \alpha_7 = 0.70, \alpha_9 = 0.55, \beta_{13} = 1.42, \beta_{15} = -1.45, \beta_{17} = -0.98, \beta_{19} = 0.52, \beta_{31} = -1.52, \beta_{35} = 0.55, \beta_{37} = 0.58, \beta_{39} = 0.62, \beta_{51} = 0.22, \beta_{53} = 0.25, \beta_{57} = 0.28, \beta_{59} = 0.32, \beta_{71} = 0.32, \beta_{73} = -1.35, \beta_{75} = 0.38, \beta_{79} = 0.42, \beta_{91} = 0.54, \beta_{93} = -1.55, \beta_{95} = 0.58, \beta_{97} = 0.64$, the equilibrium point is that $[u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*]^T = [1.3782,\ 7.6213,\ 1.1538,\ 6.1543, 1.0469,\ 5.1482, 1.1826,\ 5.3468,\ 2.1134,\ 6.1255]^T$. Therefore, we have $d_{11} = 1.7874, d_{33} = 2.0571, d_{55} = 1.9655, d_{77} = 2.2996, d_{99} = 4.0412, b_{13} = 0.4797, b_{15} = -0.7253, b_{17} = -0.4012, b_{19} = 0.0146, b_{31} = -0.5135, b_{35} = 0.2744, b_{37} = 0.2417, b_{39} = 0.0167, b_{51} = 0.0818, b_{53} = 0.1077, b_{57} = 0.1125, b_{59} = 0.0086, b_{71} = 0.1133, b_{73} = -0.5819, b_{75} = 0.1895, b_{79} = 0.0112, b_{91} = 0.1824, b_{93} = -0.6625, b_{95} = 0.2868, b_{97} = 0.2671$. The characteristic values of matrix D and B are $3.9307, 2.1198, 1.9372, 1.7865, 1.6948, 0.0198, -0.0895, -0.0274, -0.0172, -0.0011$, and $0, 0, 0, 0, 0, -0.0515, -0.0756 \pm 0.1672\, i, 0.1014 \pm 0.6593\, i$, respectively. Obviously, the conditions of Theorem 1 are satisfied. When time delays are selected as $1.85, 1.75, 1.78, 1.82, 1.80$, and $3.25, 3.15, 3.18, 3.22, 3.20,25$, respectively, there exists a periodic solution (see Fig.1 and Fig.2).

Then we keep the values of $c_i, d_i, k_i, b_i, \varepsilon_i, \alpha_i$ as in Figure 1, and only change the values of β_{ij} as $\beta_{13} = 0.62, \beta_{15} = 1.85, \beta_{17} = 1.98, \beta_{19} = -2.82, \beta_{31} = 1.18, \beta_{35} = 0.75, \beta_{37} = 0.58, \beta_{39} = 0.62, \beta_{51} = 0.28, \beta_{53} = 0.15, \beta_{57} = -0.78, \beta_{59} = 0.32, \beta_{71} = -0.76, \beta_{73} = 0.68, \beta_{75} = 0.38, \beta_{79} = 0.62, \beta_{91} = 0.57, \beta_{93} = 0.88, \beta_{95} = -0.68, \beta_{97} = 0.78$. Then the equilibrium point is changed to $[1.2058,\ 4.2832,\ 0.5438,\ 8.1944, 1.0836,\ 4.6805, 1.1072,\ 5.9301,\ 1.7148,\ 8.9704]^T$. Thus, we have $d_{11} = 1.8933, d_{33} = 1.1926, d_{55} = 2.4562, d_{77} = 2.3013, d_{99} = 4.0816, b_{13} = 0.2551, b_{15} = 0.3279, b_{17} = 0.8772, b_{19} = -0.7249, b_{31} = 0.4821, b_{35} = 0.3751, b_{37} = 0.2642, b_{39} = 0.1594, b_{51} = 0.1148, b_{53} = 0.1193\ b_{57} = -0.3529, b_{59} = 0.0823, b_{71} = -0.3108, b_{73} = 0.5357, b_{75} = 0.1739, b_{79} = 0.1594, b_{91} = 0.2331, b_{93} = 0.6875, b_{95} = 0.3412, b_{97} = 0.3529$. The characteristic values of matrix D and B are
$3.9722, 2.3151, 2.1216, 1.8061, 0.9505, 0.1021, 0.0197, -0.0906, -0.0389, -0.0328$, and
$0, 0, 0, 0, 0, -0.4400, 0.3777, 0.4992, -0.2164 \pm 0.5518\, i$, respectively. The conditions of Theorem 1 are satisfied. When time delays are selected as $2.25, 2.15, 2.18, 2.22, 2.20$, there exists a periodic solution (see Fig.3).

Then we change the amplification function as $\tanh(u_i)$. The parameters are selected as follows: $c_1 = 0.12, c_3 = 0.15, c_5 = 0.18, c_7 = 0.10, c_9 = 0.24, \alpha_1 = 0.22, \alpha_3 = 0.25, \alpha_5 = 0.20, \alpha_7 = 0.30, \alpha_9 = 0.25, \beta_{13} = 0.62, \beta_{15} = 0.85, \beta_{17} = 0.98$. The other parameters are the same as in Figure 3. We see the equilibrium point is that $[u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*]^T = [0.2506,\ 2.1036,\ 0.3728,\ 2.1362, 0.3106,\ 1.2012, 1.2508,\ 0.9826,\ 0.4716,\ 1.4506]^T$.

Thus, we have $d_{11} = 0.9275, d_{33} = 1.0093, d_{55} = 0.9395, d_{77} = 0.4849, d_{99} = 1.1296, b_{13} = 0.5412, b_{15} = 0.7731, b_{17} = 0.2744, b_{19} = -2.2748, b_{31} = 1.1089, b_{35} = 0.6821, b_{37} = 0.1642, b_{39} = 0.5011, b_{51} = 0.2631, b_{53} = 0.1411, b_{57} = -0.2184, b_{59} = 0.2581, b_{71} = -0.3462, b_{73} = 0.6390, b_{75} = 0.3456, b_{79} = 0.5012, b_{91} = 0.5367, b_{93} = 0.8271, b_{95} = -0.6188, b_{97} = 0.2184$. Therefore, $\mu(D) = 1.9596, \|B\| = 3.7536$. When time delays are selected as $0.95, 0.85, 0.88, 0.92, 0.90$, and $1.25, 1.15, 1.18, 1.22, 1.20$, respectively, then

$$\|B\| e\tau_* = 3.7536 * 0.85\ e = 8.6728 > 5.2892 = e^{1.9596*0.85} = e^{|\mu(D)|\tau_*}$$

and

$$\|B\| e\tau_* = 3.7536 * 1.15\ e = 11.7338 > 9.5214 = e^{1.9596*1.15} = e^{|\mu(D)|\tau_*}$$

The conditions of Theorem 2 are satisfied. There exists a periodic solution (see Fig.4 and Fig.5). However, when time delays are increased to $1.65, 1.55, 1.58, 1.62, 1.60$, then $\tau_* = 1.55$, and $\|B\| e\tau_* = 3.7536 * 1.55\ e = 15.8152$, in this case, $e^{|\mu(D)|\tau_*} = e^{1.9596*1.55} = 20.8505$, the conditions of Theorem 2 are not satisfied. But a periodic solution still exists (see Fig.6). This means that Theorem 2 is a stronger sufficient condition. Then we only change the parameters $\beta_{19} = -1.42, \beta_{31} = 0.18, \beta_{71} = 0.76, \beta_{73} = -0.68$, the other parameters are the same as in Figure 6, when time delays are increased to $2.95, 2.85, 2.88, 2.92, 2.90$, and $4.95, 4.85, 4.88, 4.92, 4.90$, respectively, we see that there is a periodic solution (see Fig.7 and Fig.8).

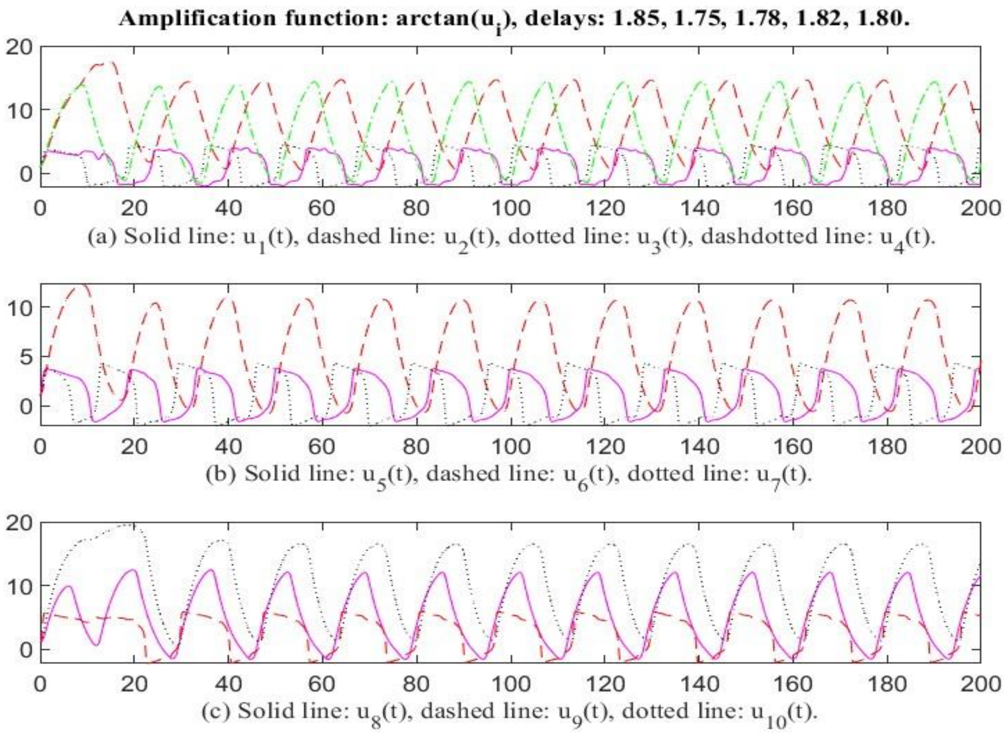

Fig.1 Oscillation of the solutions, delays: 1.85, 1.75, 1.78, 1.82, 1.80.

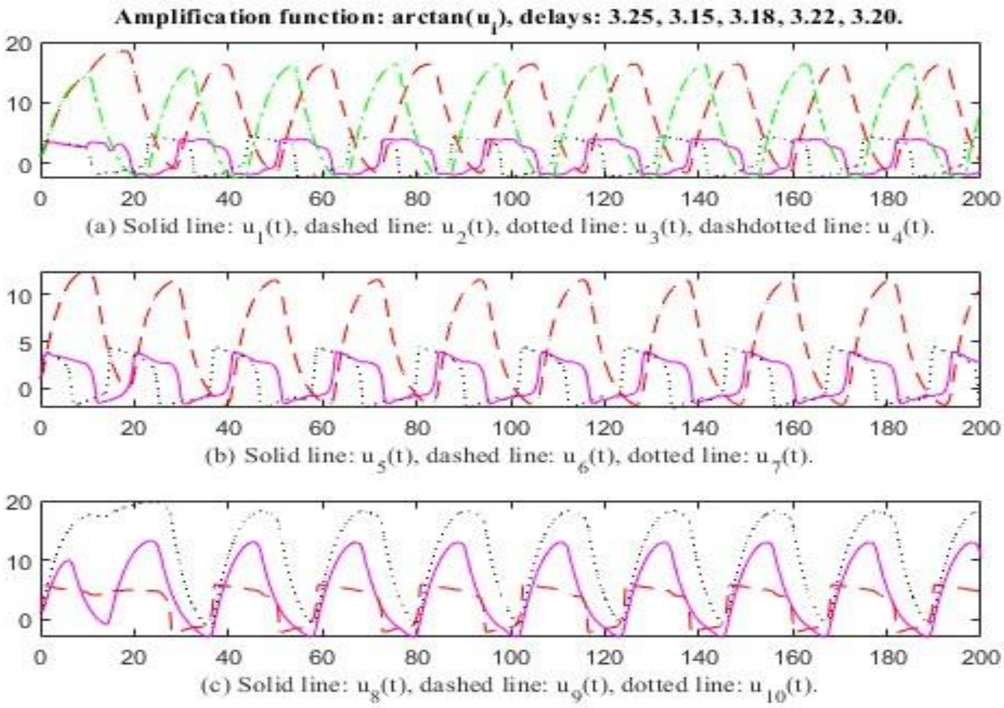

Fig.2 Oscillation of the solutions, delays: 3.25, 3.15, 3.18, 3.22, 3.20.

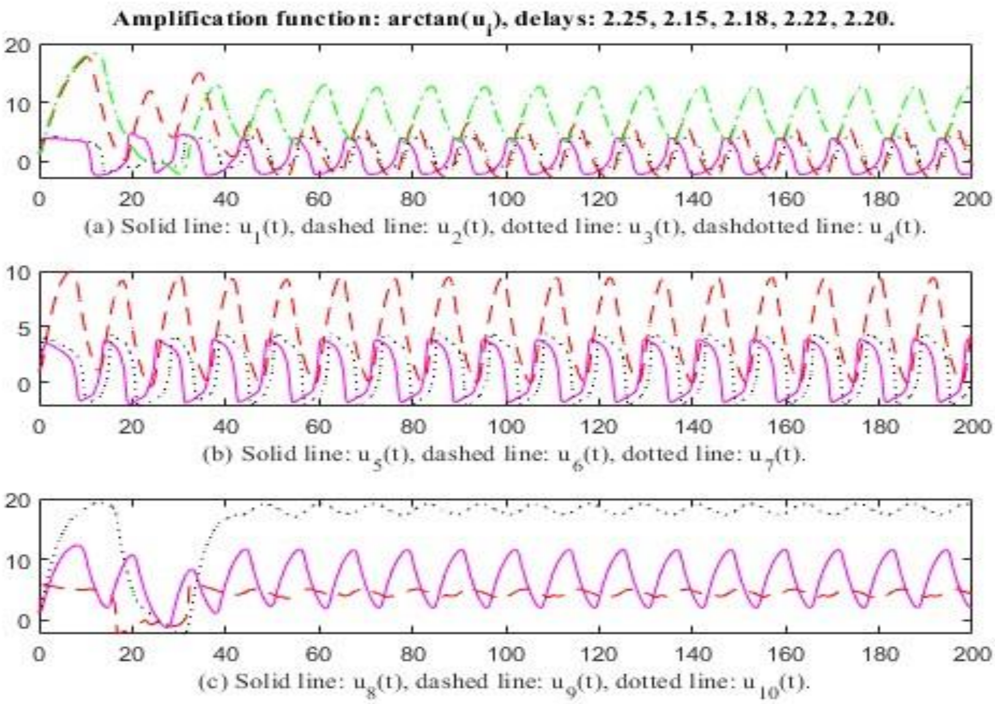

Fig.3 Oscillation of the solutions, delays: 2.25, 2.15, 2.18, 2.22, 2.20.

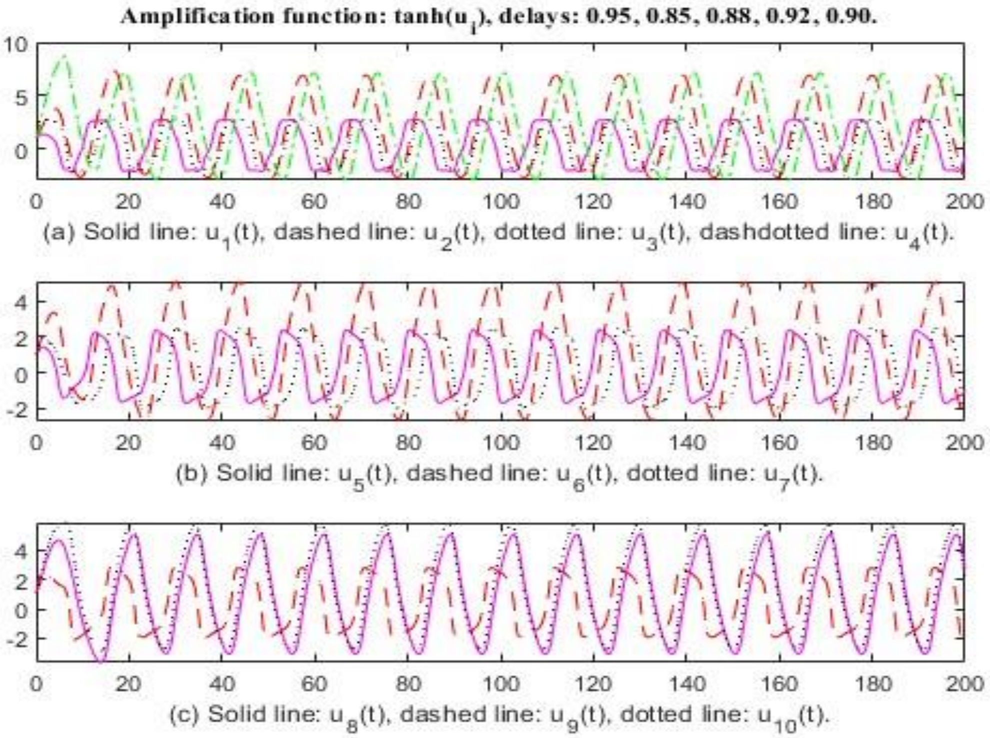

Fig.4 Oscillation of the solutions, delays: 0.95, 0.85, 0.88, 0.92, 0.90.

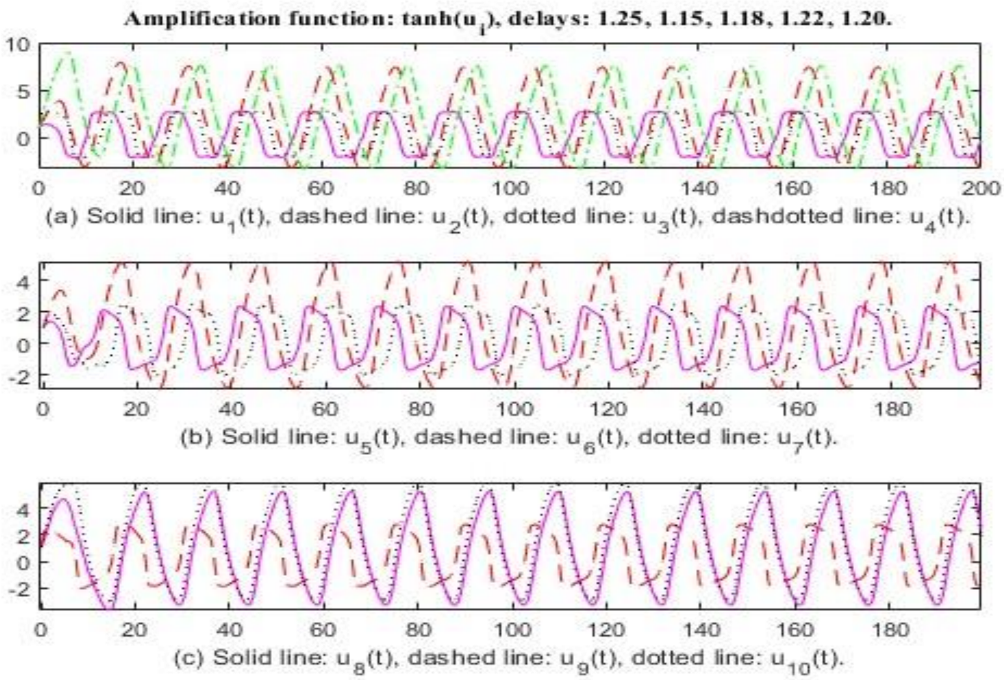

Fig.5 Oscillation of the solutions, delays: 1.25, 1.15, 1.18, 1.22, 1.20.

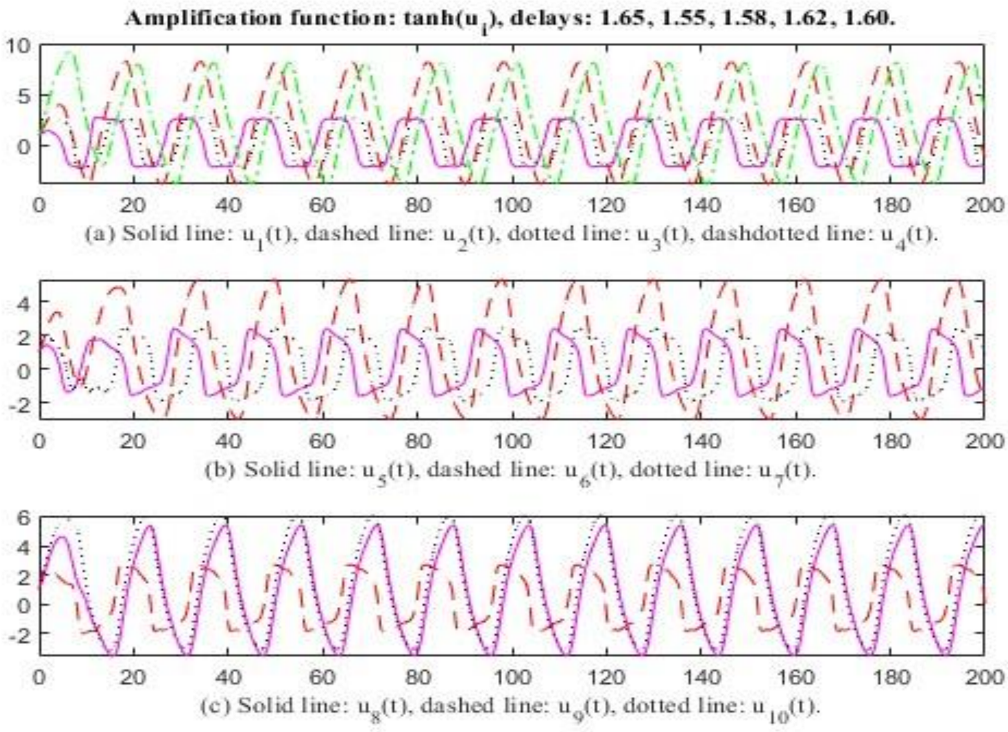

Fig.6 Oscillation of the solutions, delays: 1.65, 1.55, 1.58, 1.62, 1.60.

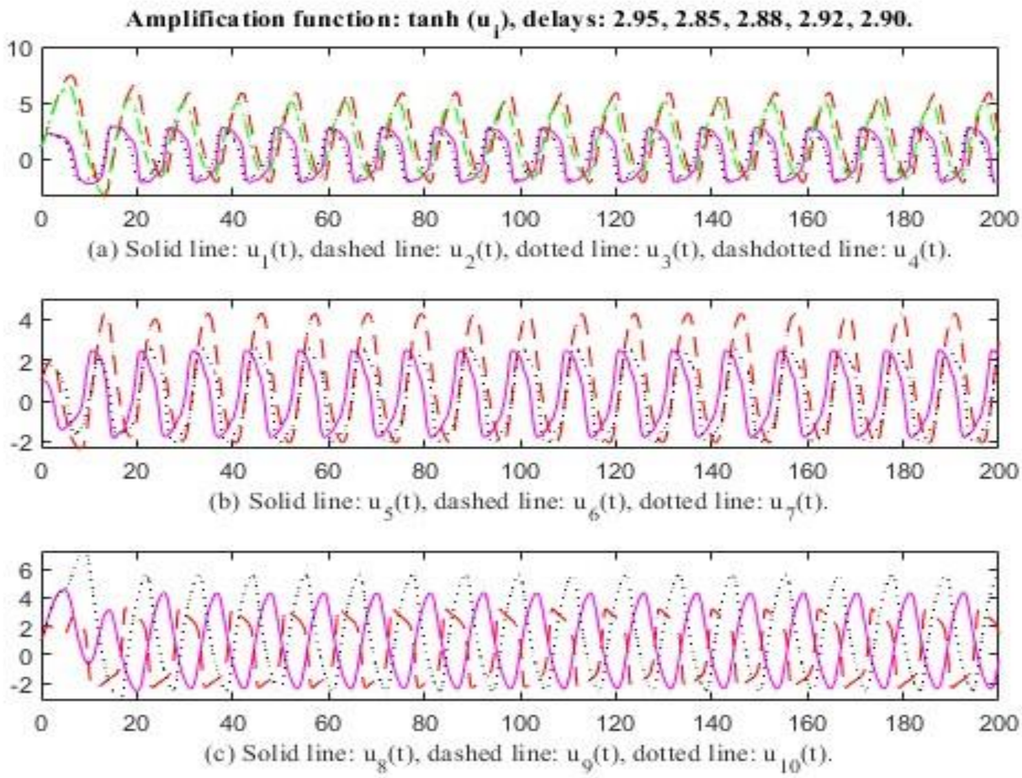

Fig.7 Oscillation of the solutions, delays: 2.95, 2.85, 2.88, 2.92, 2.90.

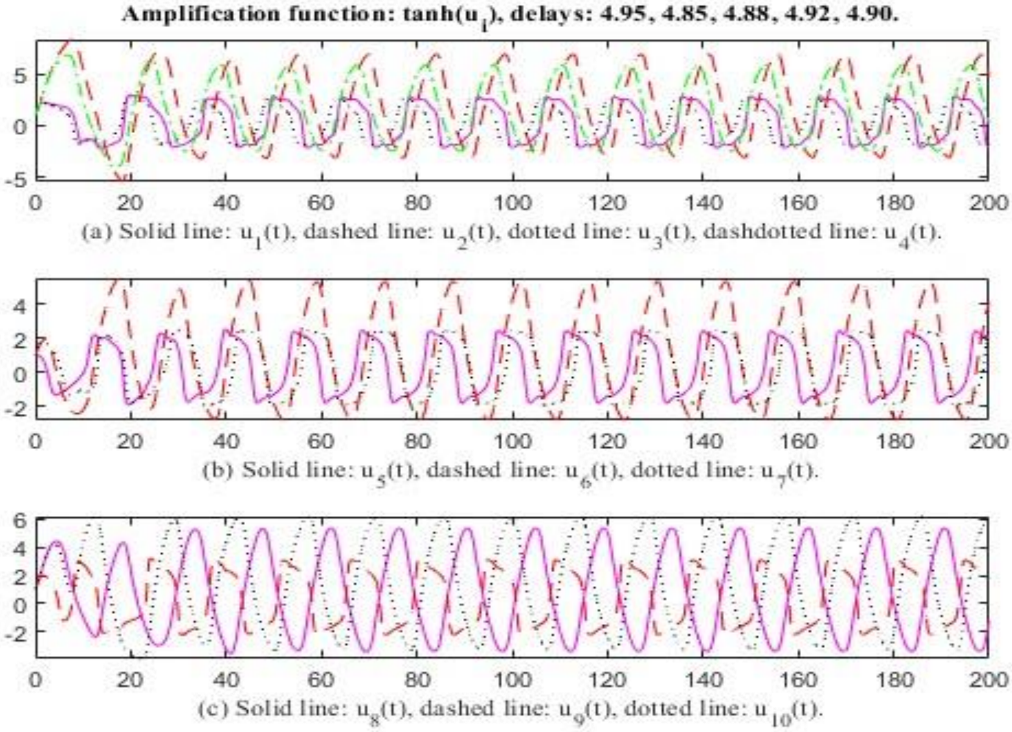

Fig.8 Oscillation of the solutions, delays: 4.95, 4.85, 4.88, 4.92, 4.90.

V. Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a five-coupled delayed FHN model with delays. Based on the method of mathematical analysis, we provided two theorems that those are only sufficient conditions to guarantee the periodic oscillation of the solutions. Some simulations are provided to indicate the effectiveness of the criteria.

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