International Journal of Modern Research in Engineering and Technology (IJMRET)
www.ijmret.org Volume 3 Issue 7 ǁ July 2018.

Modeling and vibration Analyses of a rotor having multiple disk supported by a continuous shaft for the first three modes

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July 2018g

Abstract: This paper discusses the modeling and vibration Analyses of a rotor having multiple disk supported by a continuous shaft for the first three modes. Normal modes of constrained structures method is used to develop the equations. First three modes of the beam-disk system are considered.

I. Introduction

Vibration is progressively turned to be a problem sincethe machine velocities have increased. Vibration analysis possibly will be commenced as a separate process, otherwise may perhaps be portion of a machine section inspection or full machine analysis [1]. The component of the rotor systems include, disk, blades and shaft which have been comprehensively executed in the industry. Unbalanced masses are the core sources of the vibration in rotating machinery. The impact of unbalance onrotating shafts, that is the core basis of centrifugal forces in these systems has been realized. These forces result in enormous rotation amplitudes as soon as the shaft is rotated at its ordinary frequency, which is alsoreferred to as critical speed.

The following work product will attempt to provide a brief literature review about the Saudi Electricity company, some definitions for the key words in this review, vibration analysis, Jeffcott Rotor Model, A Three-Disc Torsional Rotor System, A Jeffcott Rotor Model with an Offset Disc and will address other topics of concern.

Formulation

To illustrate the studied system a equations will be formulated by two approaches the first one is formulate and determine the general equation and first three modes of the continues shaft without masses , the second approaches that will be taken is to formulate and determine general equation and first three modes of the continues shaft combined with the masses .

This chapter describes the mathematical formulation to study the concerned system .

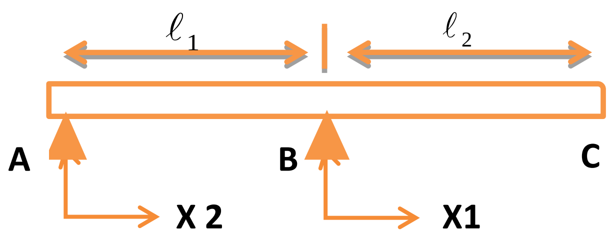

Figure (2.1)

First Approaches

To determine the natural frequencies of vibration for the concerned continuous shaft as shown in Figure (2.1) ,

(1)

identification of the boundary condition and support type at points A ,band C . point A is simply supported therefor the slope and moment is zero as illustrated in equation (1) and (2) . Point B is clamped free end where the moment and slope is also zero and the slope and moment of beam (1) (A-B) equal the slope of beam (2) (B,C) as shown in equation (3 ,4 ,5,and 6). On the other hand point C moment and sheer force for beam (2) (B,C) is zero , when ℓ_1 is zero at equation (7,8)

$$y_1(0)=0 \qquad y_1(\ell_1)=0 \qquad y_2(0)=0 \qquad y_2''(\ell_1)=0 \tag{1}$$

$$y_1''(0)=0 \qquad y_1''(\ell_1)=0 \qquad y_2''(0)=0 \qquad y_2'''(\ell_1)=0 \tag{2}$$

Additional Boundary conditions at B (Continuity conditions)

$$y_2'(0)=-y_1'(\ell_1) \tag{3}$$

$$y_2''(0)=y_1''(\ell_1) \tag{4}$$

in our study we divided the beam for two parts (1 and 2) virtually only to ease the study and understanding so the mode equation for the beam can be determined , for beam #1 Mode equation is shown below

$$y_1(x_1)=\mathrm{A}_1\cosh\beta x_1+\mathrm{B}_1\sinh\beta \mathrm{x}_1+C_1\cos\beta x_1+D_1\sin\beta x_1 \tag{5}$$

$$y_1(0)=0=\mathrm{A}_1+C_1=0 \tag{6}$$

$$y_1''(x_1)=\beta^2\left[\mathrm{A}_1\cosh\beta x_1+\mathrm{B}_1\sinh\beta \mathrm{x}_1-C_1\cos\beta x_1-D_1\sin\beta x_1\right] \tag{7}$$

$$y_1''(0)=\mathrm{A}_1-C_1=0 \tag{8}$$

$$(7) \text{ and } (8) \text{ gives } \mathrm{A}_1=0 \quad \mathrm{C}_1=0$$

After satisfying the boundary condition as we had , we found that mode equation for beam #1 is

$$y_1(x_1)=\mathrm{B}_1\sinh\beta x_1+D_1\sin\beta x_1 \tag{9}$$

and for beam #2 Mode equation is :

$$y_2(x_2)=\mathrm{A}_2\cosh\beta x_2+\mathrm{B}_2\sinh\beta \mathrm{x}_2+C_2\cos\beta x_2+D_2\sin\beta x_2 \tag{10}$$

$$y_2(0)=\mathrm{A}_2+C_2=0 \tag{11}$$

(12)

$$y_2''(0)=\beta^2\left[A_2-C_2\right]=0$$

(11) and (12) gives $A_2=0 \quad C_2=0$

$$y_2(x_2)=B_2\sinh\beta x_2+D_2\sin\beta x_2 \tag{13}$$

to get the values of B1, B2 , D1 and D2 , we needed to satisfy the additional boundary condition that we set previously . as bellow

$$y_2'(0)=-y_1'(\ell_1) \tag{14}$$

$$\beta\left(B_2\cosh\beta x_2+D_2\cos\beta x_2\right)\Big|_{x_2=0}=-\beta\left(B_1\cosh\beta x_1+D_1\sin\beta x_1\right)\Big|_{x_1=\ell_1} \tag{15}$$

$$B_2+D_2=-\left(B_1\cosh\beta\ell_1+D_1\sin\beta\ell_1\right) \tag{16}$$

then we satisfied the second boundary condition

$$y_2''(0)=-y_1''(\ell_1) \tag{17}$$

$$\beta^2\left(B_2\sinh\beta x_2-D_2\sin\beta x_2\right)\Big|_{x_1=0}=-\beta^2\left(B_1\sinh\beta x_1-D_1\sin\beta x_1\right)\Big|_{x_1=\ell_1} \tag{18}$$

$$0=-(B_1\sinh\beta\ell_1-D_1\sin\beta\ell_1)$$

for beam #1

$$y_1(\ell_1)=B_1\sinh\beta\ell_1+D_1\sin\beta\ell_1=0 \tag{19}$$

$$y_2''(\ell_1)=B_1\sinh\beta\ell_1-D_1\sin\beta\ell_1=0 \tag{20}$$

from equation (20) we obtained the next matrix .

$$\begin{vmatrix} \sinh \beta\ell_1 & \sin \beta\ell_1 \\ \sinh \beta\ell_1 & -\sin \beta\ell_1 \end{vmatrix} = 0 \qquad (21)$$

If ($2\sinh \beta\ell_1 \sin \beta\ell_1$) was equaling zero , Then ($\sinh \beta\ell_1$) can not be zero
so ($\sin \beta\ell_1$ equal zero)
Then $\left(\beta\ell_{1i}\right) = i\pi$ for $i = 1,\ldots..n$ (22)

mode functions for the beam #1 will be

$$y_{1i}\left(x_1\right) = B_i \sin(\beta\ell_1)_i \frac{x_1}{\ell_1} \qquad (23)$$

Natural frequency of beam 1 can be calculated from

$$\omega_{i2} = (\beta\ell_1)^2{}_i \sqrt{\frac{EI}{eAL_1{}^4}} \qquad (24)$$

wheres e is the denisity (kg / m^3)
and A is the shaft cros-section area (m^2)
ℓ_1 is the shaft length (M)
and E stand for Elasticity modulus ($\mathrm{N/m}^2$)
I can be discribed as momen of inrtia (m^4)
and finaly $\beta\ell_i$ represent the mode of vibartion

The deflection of the router can

From the previous equations we conclude the deflection equations for beam 1 and beam 2

And to determine the slope equations and satisfy the boundaries we should derive it as shown in the following equations (25) (26)

$$y_1(x_1) = B_1 \sinh \beta x_1 + D_1 \sin \beta x_1 \quad (25)$$

$$y_2(x_2) = B_2 \sinh \beta x_2 + D_2 \sin \beta x_2 \quad (26)$$

$$y_1'(x_1) = \beta\left(B_1 \cosh \beta x_1 + D_1 \cos \beta x_1\right) \quad (27)$$

$$y_1''(x_1) = \beta^2\left(B_1 \sinh \beta x_1 - D_1 \sin \beta x_1\right) \quad (28)$$

$$y_2'(x_2) = \beta\left(B_2 \cosh \beta x_2 + D_2 \cos \beta x_2\right) \quad (29)$$

After compensation in boundary condition in the first equation (27)

$$\begin{aligned} & y_1'(\ell_1) = -y_2'(0) \\ & \beta\left(B_1 \cosh \beta \ell_1 + D_1 \cos \beta \ell_1\right) = -\beta\left(B_2 + D_2\right) \end{aligned} \quad (29)$$

Then we also satisfy the boundary condition in the second equation (28)

$$\begin{aligned} & y_1''(\ell_1) = -y_2''(0) \\ & \beta^2\left(B_1 \sinh \beta \ell_1 - D_1 \sin \beta \ell_1\right) = -\beta^2(0-0) = 0 \end{aligned} \quad (30)$$

$$B_1 \cosh \beta \ell_1 + D_1 \cos \beta \ell_1 + B_2 + D_2 = 0 \quad (31)$$

$$B_1 \sinh \beta \ell_1 - D_1 \sin \beta \ell_1 = 0 \quad (32)$$

$$\begin{aligned} & y_2''(\ell_2) = 0 \\ & B_2 \sinh \beta \ell_2 - D_2 \sin \beta \ell_2 = 0 \end{aligned} \quad (33)$$

$$\begin{aligned} & y_2'''(\ell_2) = 0 \\ & B_2 \cosh \beta \ell_2 - D_2 \cos \beta \ell_2 = 0 \end{aligned} \quad (34)$$

To determine the beam of vibration we have to solve the previous equations in the next matrix

$$\begin{bmatrix} \cosh\beta\ell_1 & \cos\beta\ell_1 & 1 & 1 \\ \sinh\beta\ell_1 & -\sin\beta\ell_1 & 0 & 0 \\ 0 & 0 & \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ 0 & 0 & \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ D_1 \\ B_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (35)$$

$$\begin{vmatrix} \cosh\beta\ell_1 & \cos\beta\ell_1 & 1 & 1 \\ \sinh\beta\ell_1 & -\sin\beta\ell_1 & 0 & 0 \\ 0 & 0 & \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ 0 & 0 & \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{vmatrix} = 0 \quad (36)$$

open the first determinant of the matrix should be zero

$$\cosh\beta\ell_1 \begin{vmatrix} -\sin\beta\ell_1 & 0 & 0 \\ 0 & \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ 0 & \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{vmatrix} \quad (37)$$

open the second determinant of the matrix should be zero

$$-\sinh\beta\ell_1 \begin{vmatrix} \cos\beta\ell_1 & 1 & 1 \\ 0 & \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ 0 & \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{vmatrix} = 0 \quad (38)$$

$$\cosh\beta\ell_1\left(-\sin\beta\ell_1\right) \begin{vmatrix} \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{vmatrix} \quad (39)$$

$$-\sinh\beta\ell_1 \cos\beta\ell_1 \begin{vmatrix} \sinh\beta\ell_2 & -\sin\beta\ell_2 \\ \cosh\beta\ell_2 & -\cos\beta\ell_2 \end{vmatrix} = 0 \quad (40)$$

characteristic equation

$$-\left(\cosh\beta\ell_1 \sin\beta\ell_1 + \sinh\beta\ell_1 \cos\beta\ell_1\right)\left(-\sinh\beta\ell_2 \cos\beta\ell_2 + \cosh\beta\ell_2 \sin\beta\ell_2\right) = 0 \quad (41)$$

If $\ell_1 = \ell_2 = \ell$

$$\left(\cosh\beta\ell \sin\beta\ell\right)^2 - \left(\sinh\beta\ell \cos\beta\ell\right)^2 = 0 \quad (42)$$

Or

$$\tan^2\beta\ell - \tanh^2\beta\ell = 0 \quad (43)$$

if $\ell_1 = \ell \quad \ell_2 = \dfrac{\ell}{2}$

$$(\cosh\beta\ell\sin\beta\ell + \sinh\beta\ell\cos\beta\ell)(\cosh\beta\frac{\ell}{2}\sin\beta\frac{\ell}{2} - \sinh\beta\frac{\ell}{2}\cos\beta\frac{\ell}{2}) = 0 \qquad (44)$$

First three moods are :

$(\beta\ell)_1 = 3.9266$

$(\beta\ell)_2 = 7.0686$

$(\beta\ell)_3 = 10,2102$

.

$$y_1(x_1) = B_1\sinh\beta x_1 + D_1\sin\beta x_1 \qquad (45)$$

$$y_2(x_2) = B_2\sinh\beta x_2 + D_2\sin\beta x_2 \qquad (46)$$

$$y_2''(\ell_2) = 0 = \beta^2(B_2\sinh\beta\ell_2 - D_2\sin\beta\ell_2) = 0 \qquad (47)$$

$$D_2 = \frac{\sinh\beta\ell_2}{\sin\beta\ell_2} \qquad (48)$$

$$y_2(x_2) = B_2\left[\sinh\beta x_2 + \frac{\sinh\beta\ell_2}{\sin\beta\ell_2}\sin\beta x_2\right] \qquad (49)$$

$$\mathrm{y}_{2i}(x_2) = B_2\left[\sinh(\beta\ell_2)_i\frac{x_2}{\ell_2} + \frac{\sinh(\beta\ell_2)_i}{\sin(\beta\ell_2)_i}\sin(\beta\ell_2)_i\frac{x_2}{\ell_2}\right] \qquad 0<\frac{x_2}{\ell_2}<1 \qquad (50)$$

The dynamic equation of the motion is gained through methods of motion mode ,and deflection of the beam can be written as

mode summation assumption

$$\mathrm{y}_1(x_1,t) = \sum_{i=1}^{n}\Phi_{1i}(x_1)q_{1i}(t) \qquad (51)$$

$$\mathrm{y}_2(x_2,t) = \sum_{i=1}^{n}\Phi_{2i}(x_2)q_{2i}(t) \qquad (52)$$

Wheres $\Phi(x)$ is the mode function and q(t) is generalised coordinate

Function of beam mode can be expressed as

$$\Phi_{1i}(x) = D_1 \sin(\beta\ell_1)_i \frac{x_1}{\ell_1} \qquad 0 < \frac{x_1}{\ell_1} < 1 \qquad (53)$$

$$\Phi_{2i}(x) = B_2 \left[\sinh(\beta\ell_2)_i \frac{x_2}{\ell_2} + \frac{\sinh(\beta\ell_2)_i}{\sin(\beta\ell_2)_i} \sin(\beta\ell_2)_i \frac{x_2}{\ell_2} \right] \qquad 0 < \frac{x_2}{\ell_2} < 1 \qquad (54)$$

Continues shaft and disk model

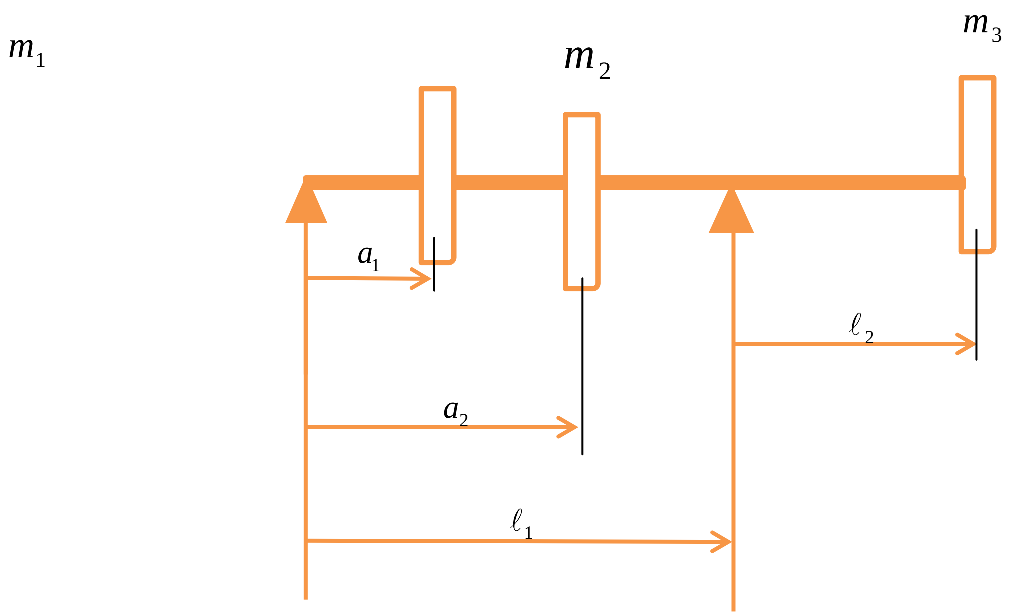

Figure (2.2) Continues shaft and disk model

Three disks with masses m1, m2 and m3 are added to the continuous shaft location a1 , a2 and L2 .

Whereas a1 is defined as the distance between bearing (1) and m1 , while a2 is the distance between bearing (2) and m2 , leading intuitively that L1 is the distance between bearing (1) and bearing (2) , finally L2 is the distance between bearing (2) and m3

Second Approaches

The second approaches that will be taken is to formulate and determine general equation and first three modes of the continues shaft combined with the masses .

This approaches lead us to a knowledge about the results after combining it with the masses , which will be studied in details in chapter ( ) , furthermore the difference between the frequencies and its effect on the mode after adding the masses will also be studied

If we decided to consider viscous damper and the beam , then equation of motion is going to be

$$\ddot{q}_i + 2\zeta\omega_i\dot{q}_i + \omega_i^2 q_i \tag{55}$$

$$= \frac{1}{M_{ii}}\left[\int f_1(x,t)\Phi_{1i}(x)dx + \int f_2(x,t)\Phi_{2i}(x)dx\right] \tag{56}$$

$$= \frac{1}{M_{ii}}\left[f_1(a_1,t)\Phi_{1i}(a_1) + \int f_1(a_2,t)\Phi_{1i}(a_2) + f_2(\ell_2,t)\Phi_{2i}(\ell_2)\right] \tag{57}$$

$$= \frac{1}{M_{ii}}\left[-m_1\ddot{y}_1(a_1,t)\Phi_{1i}(a_1) - m_2\ddot{y}_1(a_2,t)\Phi_{1i}(a_2) - m_3\ddot{y}_2(\ell_2,t)\Phi_{2i}(\ell_2)\right] \tag{58}$$

Generalized mass can be expressed as

$$M_{ii} = \int_0^{\ell_1} m(x_1)^2\,\Phi_{1i}(x)dx \qquad m(x_1) \text{ is the mass of the shaft per unit length} \tag{59}$$

$$\ddot{y}_1 = \sum_{i=1}^{3}\ddot{q}_i(\mathrm{t})\Phi_{1i}(\mathrm{x}) = q_1(\mathrm{t})\Phi_{11}(\mathrm{x}) + \ddot{q}_2(\mathrm{t})\Phi_{12}(\mathrm{x}) + \ddot{q}_3(\mathrm{t})\Phi_{13}(\mathrm{x}) \tag{60}$$

$$= \frac{1}{M_{ii}}\left[-m_1\left(\sum_{J=1}^{3}\ddot{q}_J\Phi_{1J}(a_1)\right)\Phi_{1i}(a_1) - m_2\left(\sum_{J=1}^{3}\ddot{q}_J\Phi_{1J}(a_2)\right)\Phi_{1i}(a_2) - m_3\left(\sum_{J=1}^{3}\ddot{q}_J\Phi_{2J}(\ell_2)\right)\Phi_{2i}(\ell_2)\right] \tag{61}$$

$$\begin{aligned} &= \frac{1}{M_{ii}}\left[-m_1\left(\ddot{q}_1\Phi_{11}(a_1) + \ddot{q}_2\Phi_{12}(a_1) + \ddot{q}_3\Phi_{13}(a_1)\right)\right]\Phi_{1i}(a_1) \\ &-m_2\left[\left(\ddot{q}_1\Phi_{11}(a_2) + \ddot{q}_2\Phi_{12}(a_2) + \ddot{q}_3\Phi_{13}(a_2)\right)\right]\Phi_{1i}(a_2) \\ &-m_3\left[\left(\ddot{q}_1\Phi_{21}(\ell_2) + \ddot{q}_2\Phi_{22}(\ell_2) + \ddot{q}_3\Phi_{23}(\ell_2)\right)\right]\Phi_{2i}(\ell_2) \end{aligned} \tag{62}$$

The generalized equation of motion for the first mode

$i = 1$

$$\ddot{q}_1 + 2\zeta\omega_1\dot{q}_1 + \omega_1^2 q_1 \tag{63}$$

$$\begin{aligned} &= \frac{1}{M_{ii}}\left[-m_1\left(\ddot{q}_1\Phi_{11}(a_1)\Phi_{11}(a_1) + \ddot{q}_2\Phi_{12}(a_1)\Phi_{11}(a_1) + \ddot{q}_3\Phi_{13}(a_1)\Phi_{11}(a_1)\right)\right. \\ &-m_2\left(\ddot{q}_1\Phi_{11}(a_2)\Phi_{11}(a_2) + \ddot{q}_2\Phi_{12}(a_2)\Phi_{11}(a_2) + \ddot{q}_3\Phi_{13}(a_2)\Phi_{11}(a_2)\right) \\ &\left.-m_3\left(\ddot{q}_1\Phi_{21}(\ell_2)\Phi_{21}(\ell_2) + \ddot{q}_2\Phi_{21}(\ell_2)\Phi_{21}(\ell_2) + \ddot{q}_3\Phi_{23}(\ell_2)\Phi_{21}(\ell_2)\right)\right] \end{aligned} \tag{64}$$

$$\ddot{q}_1\left[1+m_1\Phi_{11}(a_1)\Phi_{11}(a_1)+m_{11}\Phi_{11}(a_2)\Phi_{11}(a_2)+M_3\Phi_{21}(\ell_2)\Phi_{21}(\ell_2)\right]\frac{1}{M_{11}}$$
$$+\ddot{q}_2\left[m_1\Phi_{12}(a_1)\Phi_{11}(a_1)+m_{12}\Phi_{12}(a_2)\Phi_{11}(a_2)+M_3\Phi_{22}(\ell_2)\Phi_{21}(\ell_2)\right]\frac{1}{M_{11}}$$
$$+\ddot{q}_3\left[m_1\Phi_{13}(a_1)\Phi_{11}(a_1)+m_{13}\Phi_{13}(a_2)\Phi_{11}(a_2)+M_3\Phi_{23}(\ell_2)\Phi_{21}(\ell_2)\right]\frac{1}{M_{11}}$$
$$+2\zeta\omega_1\dot{q}_1+\omega_1^2q_1=0 \quad (65)$$

The generalized equation of motion for the second mode

$$i=2 \quad (66)$$
$$\ddot{q}_2+2\zeta\omega_2\dot{q}_2+\omega_1^2q_2$$
$$=\frac{1}{M_{22}}\left[-m_1\left(\ddot{q}_1\Phi_{11}(a_1)\Phi_{12}(a_1)+\ddot{q}_2\Phi_{12}(a_1)\Phi_{12}(a_1)+\ddot{q}_3\Phi_{13}(a_1)\Phi_{12}(a_1)\right)\right.$$
$$-m_2\left(\ddot{q}_1\Phi_{11}(a_2)\Phi_{12}(a_2)+\ddot{q}_2\Phi_{12}(a_2)\Phi_{12}(a_2)+\ddot{q}_3\Phi_{13}(a_2)\Phi_{12}(a_2)\right) \quad (67)$$
$$\left.-m_3\left(\ddot{q}_1\Phi_{21}(\ell_2)\Phi_{22}(\ell_2)+\ddot{q}_2\Phi_{22}(\ell_2)\Phi_{22}(\ell_2)+\ddot{q}_3\Phi_{23}(\ell_2)\Phi_{22}(\ell_2)\right)\right]$$

$$\ddot{q}_1\left[m_1\Phi_{11}(a_1)\Phi_{12}(a_1)+m_{21}\Phi_{11}(a_2)\Phi_{12}(a_2)+m_3\Phi_{21}(\ell_2)\Phi_{22}(\ell_2)\right]\frac{1}{M_{22}} \quad (68)$$
$$+\ddot{q}_2\left[1+m_1\Phi_{12}(a_1)\Phi_{12}(a_1)+m_{22}\Phi_{12}(a_2)\Phi_{12}(a_2)+m_3\Phi_{22}(\ell_2)\Phi_{22}(\ell_2)\right]\frac{1}{M_{22}}$$
$$+\ddot{q}_3\left[m_1\Phi_{13}(a_1)\Phi_{12}(a_1)+m_{23}\Phi_{13}(a_2)\Phi_{12}(a_2)+M_3\Phi_{23}(\ell_2)\Phi_{22}(\ell_2)\right]\frac{1}{M_{22}}$$
$$+2\zeta\omega_2\dot{q}_2+\omega_2^2q_2=0$$

The generalized equation of motion for the third mode

$$i=3$$
$$\ddot{q}_1\left[m_1\Phi_{11}(a_1)\Phi_{13}(a_1)+m_{31}\Phi_{11}(a_2)\Phi_{13}(a_2)+m_3\Phi_{21}(\ell_2)\Phi_{23}(\ell_2)\right]\frac{1}{M_{33}} \quad (69)$$
$$+\ddot{q}_2\left[m_1\Phi_{12}(a_1)\Phi_{13}(a_1)+m_{32}\Phi_{12}(a_2)\Phi_{13}(a_2)+m_3\Phi_{22}(\ell_2)\Phi_{23}(\ell_2)\right]\frac{1}{M_{33}}$$
$$+\ddot{q}_3\left[1+m_1\Phi_{13}(a_1)\Phi_{13}(a_1)+m_{33}\Phi_{13}(a_2)\Phi_{13}(a_2)+M_3\Phi_{23}(\ell_2)\Phi_{23}(\ell_2)\right]\frac{1}{M_{33}}$$
$$+2\zeta\omega_3\dot{q}_3+\omega_3^2q_3=0$$

Now we can express the equations of motion for first three modes of the beam in second approaches as below sequencely :

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + m_{13}\ddot{q}_3 + 2\zeta\omega_1\dot{q}_1 + \omega_1^2 q_1 = 0$$
$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + m_{23}\ddot{q}_3 + 2\zeta\omega_2\dot{q}_2 + \omega_2^2 q_2 = 0 \qquad (70)$$
$$m_{31}\ddot{q}_1 + m_{32}\ddot{q}_2 + m_{33}\ddot{q}_3 + 2\zeta\omega_3\dot{q}_3 + \omega_3^2 q_3 = 0$$

Also can be written in a matrix form as follows;

$$\begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} 2\zeta\omega_1 & 0 & 0 \\ 0 & 2\zeta\omega_2 & 0 \\ 0 & 0 & 2\zeta\omega_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (71)$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = M \qquad \text{mass matrix} \qquad (72)$$

$$\begin{bmatrix} 2\zeta\omega_1 & 0 & 0 \\ 0 & 2\zeta\omega_2 & 0 \\ 0 & 0 & 2\zeta\omega_3 \end{bmatrix} = C \qquad \text{damping matrix} \qquad (73)$$

$$\begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} = K \qquad \text{shiftness matrix} \qquad (74)$$

equation (75) is written in a closed form

$$M\ddot{q} + C\dot{q} + Kq = 0 \qquad \text{equation of the motion} \qquad (75)$$

The mass matrix elements are illustrated below

$$m_{11} = \ddot{q}_1 \left[1 + m_1 \Phi_{11}(a_1) \Phi_{11}(a_1) + m_2 \Phi_{11}(a_2) \Phi_{11}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{21}(\ell_2) \right] \frac{1}{M_{11}} \quad (76)$$

$$m_{12} = \ddot{q}_1 \left[m_1 \Phi_{11}(a_1) \Phi_{12}(a_1) + m_2 \Phi_{11}(a_2) \Phi_{12}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{22}(\ell_2) \right] \frac{1}{M_{11}} \quad (77)$$

$$m_{13} = \ddot{q}_1 \left[m_1 \Phi_{11}(a_1) \Phi_{13}(a_1) + m_2 \Phi_{11}(a_2) \Phi_{13}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{23}(\ell_2) \right] \frac{1}{M_{11}} \quad (78)$$

$$m_{21} = \ddot{q}_2 \left[m_1 \Phi_{12}(a_1) \Phi_{11}(a_1) + m_2 \Phi_{12}(a_2) \Phi_{11}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{21}(\ell_2) \right] \frac{1}{M_{22}} \quad (79)$$

$$m_{22} = \ddot{q}_2 \left[1 + m_1 \Phi_{12}(a_1) \Phi_{12}(a_1) + m_2 \Phi_{12}(a_2) \Phi_{12}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{22}(\ell_2) \right] \frac{1}{M_{22}} \quad (80)$$

$$m_{23} = \ddot{q}_2 \left[m_1 \Phi_{12}(a_1) \Phi_{13}(a_1) + m_2 \Phi_{12}(a_2) \Phi_{13}(a_2) + m_3 \Phi_{21}(\ell_2) \Phi_{23}(\ell_2) \right] \frac{1}{M_{22}} \quad (81)$$

$$m_{31} = \ddot{q}_3 \left[m_1 \Phi_{13}(a_1) \Phi_{11}(a_1) + m_2 \Phi_{13}(a_2) \Phi_{11}(a_2) + m_3 \Phi_{23}(\ell_2) \Phi_{21}(\ell_2) \right] \frac{1}{M_{33}} \quad (82)$$

$$m_{32} = \ddot{q}_3 \left[m_1 \Phi_{13}(a_1) \Phi_{12}(a_1) + m_2 \Phi_{13}(a_2) \Phi_{12}(a_2) + m_3 \Phi_{23}(\ell_2) \Phi_{22}(\ell_2) \right] \frac{1}{M_{33}} \quad (83)$$

$$m_{33} = \ddot{q}_3 1 + \left[m_1 \Phi_{13}(a_1) \Phi_{13}(a_1) + m_2 \Phi_{13}(a_2) \Phi_{13}(a_2) + m_3 \Phi_{23}(\ell_2) \Phi_{23}(\ell_2) \right] \frac{1}{M_{33}} \quad (84)$$

In this point we change the equation ( ) into state space equation as following :

$$\dot{q}=I\dot{q}$$
$$\ddot{q}=-M^{-1}Kq-M^{-1}C\dot{q}$$
$$\begin{bmatrix}\dot{q}\\ \ddot{q}\end{bmatrix}=\begin{bmatrix}O & I\\ -M^{-1}K & -M^{-1}C\end{bmatrix}\begin{bmatrix}q\\ \dot{q}\end{bmatrix} \quad (85)$$

$$\Phi_{1i}(x_1)=B_1\left[\sinh(\beta\ell)_i\frac{x_1}{\ell}-\frac{\sinh(\beta\ell)_i}{\sin(\beta\ell)_i}\sin(\beta\ell)_i\frac{x_1}{\ell}\right] \quad (86)$$

$$\Phi_{2i}(x_2)=B_2\left[\sinh(\beta\ell)_i\frac{x_2}{\ell}+\frac{\sinh(\beta\ell)_i}{\sin(\beta\ell)_i}\sin(\beta\ell)_i\frac{x_2}{\ell}\right] \quad (87)$$

$$\Phi_{11}(a_1)=B_1\left[\sinh(\beta\ell)_1\frac{a_1}{\ell}-\frac{\sinh(\beta\ell)_1}{\sin(\beta\ell)_1}\sin(\beta\ell)_1\frac{a_1}{\ell}\right] \quad (88)$$

$$\Phi_{11}(a_2)=B_1\left[\sinh(\beta\ell)_1\frac{a_2}{\ell}-\frac{\sinh(\beta\ell)_1}{\sin(\beta\ell)_1}\sin(\beta\ell)_1\frac{a_2}{\ell}\right] \quad (89)$$

$$\Phi_{12}(a_1)=B_1\left[\sinh(\beta\ell)_2\frac{a_1}{\ell}-\frac{\sinh(\beta\ell)_2}{\sin(\beta\ell)_2}\sin(\beta\ell)_2\frac{a_1}{\ell}\right] \quad (90)$$

$$\Phi_{12}(a_2)=B_1\left[\sinh(\beta\ell)_2\frac{a_2}{\ell}-\frac{\sinh(\beta\ell)_2}{\sin(\beta\ell)_2}\sin(\beta\ell)_2\frac{a_2}{\ell}\right] \quad (91)$$

$$\Phi_{13}(a_1)=B_1\left[\sinh(\beta\ell)_3\frac{a_1}{\ell}-\frac{\sinh(\beta\ell)_3}{\sin(\beta\ell)_3}\sin(\beta\ell)_3\frac{a_1}{\ell}\right] \quad (92)$$

$$\Phi_{13}(a_2)=B_1\left[\sinh(\beta\ell)_3\frac{a_2}{\ell}-\frac{\sinh(\beta\ell)_3}{\sin(\beta\ell)_3}\sin(\beta\ell)_3\frac{a_2}{\ell}\right] \quad (93)$$

$$\Phi_{21}(\ell_2)=B_1\left[\sinh(\beta\ell)_1\frac{\ell_2}{\ell}+\frac{\sinh(\beta\ell)_1}{\sin(\beta\ell)_1}\sin(\beta\ell)_1\frac{\ell_2}{\ell}\right] \quad (94)$$

$$\Phi_{22}(\ell_2)=B_1\left[\sinh(\beta\ell)_2\frac{\ell_2}{\ell}+\frac{\sinh(\beta\ell)_2}{\sin(\beta\ell)_2}\sin(\beta\ell)_2\frac{\ell_2}{\ell}\right] \quad (95)$$

$$\Phi_{23}(\ell_2)=B_1\left[\sinh(\beta\ell)_3\frac{\ell_2}{\ell}+\frac{\sinh(\beta\ell)_3}{\sin(\beta\ell)_3}\sin(\beta\ell)_3\frac{\ell_2}{\ell}\right] \quad (96)$$

The following determinant should be zero

$$\begin{vmatrix}O & I\\ -M^{-1}K & -M^{-1}C\end{vmatrix}=0 \quad (97)$$

Numerical solution of the determinant gives Eigen values of the equation

Eigen values are

$$\lambda_i = -\zeta_i\omega_i \pm \omega_i\sqrt{1-\zeta^2{}_i}\, j = -\sigma \pm \omega dj \tag{98}$$

$$\zeta_i\omega_i = \sigma$$
$$\omega_i\sqrt{1-\zeta^2{}_i} = \omega_d$$
$$\zeta^2{}_i\omega^2{}_i = \sigma^2$$
$$\omega^2{}_i(1-\zeta^2{}_i) = \omega^2{}_d$$
$$\omega^2{}_i - \omega_i\zeta^2{}_i = \omega^2{}_d$$
$$\omega^2{}_i = \sigma^2 + \omega^2{}_d$$

Natural frequencies and damping ratio of beam which is studied can be calculated from;

$$\omega_i = \sqrt{\sigma^2 + \omega^2{}_d} \quad \text{natural frequency} \tag{99}$$

$$\zeta_i = \frac{\sigma}{\omega_i} = \frac{\sigma}{\sqrt{\sigma^2 + \omega^2{}_d}} \quad \text{damping ratio} \tag{100}$$

II. RESULTS

To illustrate the studied system a equations will be formulated by two approaches the first one is formulate and determine the general equation and first three modes of the continues shaft without masses , the second approaches that will be taken is to formulate and determine general equation and first three modes of the continues shaft combined with the masses , then we solve the equations for the two approaches using Matlab .

1- We created a program to solve the equations

2- The parameters that we written in the program are as following :

Parameter	**Value**
Elasticity modulus (E)	$E = 207\times10^9$ N/m^2
Mass density (ro)	ro=7700 kg/m^3
Shaft diameter (d)	d=0.25 m
Moment of inertia (mI)	mI=pi$\times$d^4 / 64
Viscous damping ratio (z)	Z=0.02
A	A=pi$\times$d^2 / 4

case 1

Parameter	**Value**
Elasticity modulus (E)	$E = 207 \times 10^9$ N/m^2
Mass density (ro)	ro=7700 kg/m^3
Shaft diameter (d)	d=0.25 m
Moment of inertia (mI)	mI=pi$\times$d^4 / 64
Viscous damping ratio (z)	Z=0.02
A	A=pi$\times$d^2 / 4
m1=m2=m3	= 30 kg
L1	= 1 m
L2	=1 m
a1	= ,33 × L1 m
a2	= ,66× L1 m
Natural frequencies	
Natural frequency 1 without mass (w1)	= 1.8125× 10 ^ 3
Natural frequency 2 without mass (w2)	= 3.3782 × 10 ^ 4
Natural frequency 3 without mass (w3)	= 5.7769 × 10 ^ 4
Natural frequency 1 with mass (mw1)	= 0.3190
Natural frequency 2 with mass (mw2)	= 1.7298 × 10 ^ 3
Natural frequency 3 with mass (mw3)	= 3.3563 × 10 ^ 4
r21	= 3.9266
r22	= 7.0686
r23	= 10.2102

case 2

Parameter	**Value**
Elasticity modulus (E)	$E = 207 \times 10^9$ N/m^2
Mass density (ro)	ro=7700 kg/m^3
Shaft diameter (d)	d=0.25 m
Moment of inertia (mI)	mI=pi$\times$d^4 / 64
Viscous damping ratio (z)	Z=0.02
A	A=pi$\times$d^2 / 4
m1	= 90 kg
m2	= 90 kg
m3	= 40 kg
L1	= 1 m
L2	=1 m
a1	= ,33 × L1 m
a2	= ,66× L1 m
Natural frequencies	
Natural frequency 1 without mass (w1)	= 1.8125× 10 ^ 3
Natural frequency 2 without mass (w2)	= 3.3782 × 10 ^ 4
Natural frequency 3 without mass (w3)	= 5.7769 × 10 ^ 4
Natural frequency 1 with mass (mw1)	= 0.2763
Natural frequency 2 with mass (mw2)	= 1.5935 × 10 ^ 3
Natural frequency 3 with mass (mw3)	= 3.3052 × 10 ^ 4
r21	= 7.0686
r22	= 10.2102
r23	= 13.3518

case 3

Parameter	**Value**
Elasticity modulus (E)	$E = 207 \times 10^9$ N/m^2
Mass density (ro)	ro=7700 kg/m^3
Shaft diameter (d)	d=0.25 m
Moment of inertia (mI)	mI=pi$\times$d^4 / 64
Viscous damping ratio (z)	Z=0.02
A	A=pi$\times$d^2 / 4
m1 = m2= m3	= 30 kg
L1	= 2 m
L2	= 2 m
a1	= 0,25 × L1 m
a2	= 0,75 × L1 m
Natural frequencies	
Natural frequency 1 without mass (w1)	= 453.1371
Natural frequency 2 without mass (w2)	= 8.4455 × 10 ^ 3
Natural frequency 3 without mass (w3)	= 1.4442 × 10 ^ 4
Natural frequency 1 with mass (mw1)	= 1.9738 × 10 ^ -6
Natural frequency 2 with mass (mw2)	= 445.8023
Natural frequency 3 with mass (mw3)	= 9.5655 × 10 ^ 3
r21	= 7.0686
r22	= 10.2102
r23	= 13,3518

case 4

Parameter	**Value**
Elasticity modulus (E)	$E = 207 \times 10^9$ N/m^2
Mass density (ro)	ro=7700 kg/m^3
Shaft diameter (d)	d=0.25 m
Moment of inertia (mI)	mI=pi$\times$d^4 / 64
Viscous damping ratio (z)	Z=0.02
A	A=pi$\times$d^2 / 4
m1	= 90 kg
m2	= 90 kg
m3	= 40 kg
L1	= 2 m
L2	= 2 m
a1	= 0,25 × L1 m
a2	= 0,75 × L1 m
Natural frequencies	
Natural frequency 1 without mass (w1)	= 453.1371
Natural frequency 2 without mass (w2)	= 8.4455 × 10 ^ 3
Natural frequency 3 without mass (w3)	= 1.4442 × 10 ^ 4
Natural frequency 1 with mass (mw1)	= 1.8223 × 10 ^ -6
Natural frequency 2 with mass (mw2)	= 432.1374
Natural frequency 3 with mass (mw3)	= 9.9846 × 10 ^ 3
r21	= 7.0686
r22	= 10.2102
r23	= 13.3518

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