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AN ENTIRE FUNCTION SHARING A LINEAR POLYNOMIAL WITH ITS LINEAR DIFFERENTIAL POLYNOMIALS.

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Abstract. In this paper we study the uniqueness of an entire function when it shares a linear polynomial with its linear differential polynomials.

1. Introduction, Definitions and Results

Let f be a noncostant meromorphic function defined in the open complex plane $\mathbb{C}$. The integrated counting function of poles of f is defined by

$$N(r,\infty;f)=\int_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r,$$

where $n(t,\infty;f)$ be the number of poles of f lying in $|z|\leq r$, the poles are counted according to their multiplicities and $n(0,\infty;f)$ be the multiplicity of pole of f at origin.

For a polynomial $a=a(z)$, $N(r,a;f)\,(\overline{N}(r,a;f))$ be the integrated counting function (reduced counting function) of zeros of $f-a$ in $|z|\leq r$.

Let $A\subset\mathbb{C}$, we denote by $n_A(r,a;f)$ the number of zeros of $f-a$, counted with multiplicities, that lie in $\{z:|z|\leq r\}\cap A$. The corresponding integrated counting function $N_A(r,a;f)$ is defined by

$$N_A(r,a;f)=\int_0^r \frac{n_A(t,a;f)-n_A(0,a;f)}{t}dt+n_A(0,a;f)\log r.$$

We also denote by $\overline{N}_A(r,a;f)$ the reduced counting functions of those zeros of $f-a$ that lie in $\{z:|z|\leq r\}\cap A$.

Clearly if $A=\mathbb{C}$, then $N_A(r,a;f)=N(r,a;f)$ and $\overline{N}_A(r,a;f)=\overline{N}(r,a;f)$.

We denote by $E(a,f)$ the set of zeros of $f-a$ counted with multiplicities and by $\overline{E}(a,f)$ the set of distinct zeros of $f-a$.

For the standard definitions and notations of the value distribution theory authors suggest to see [1] and [8]

The investigation of uniqueness of an entire function sharing certain values with its derivatives was initiated by L. A. Rubel and C. C. Yang [7] in 1977. They proved the following result.

Theorem A. [7]. *Let f be a non-constant entire function. If $E(a;f)=E(a;f^{(1)})$ and $E(b;f)=E(b;f^{(1)})$, for distinct finite complex numbers a and b, then $f\equiv f^{(1)}$.*

In 1979 E.Mues and N.Steinmetz [6] took up the case of IM sharing in the place of CM sharing of values and proved the following theorem.

Theorem B. [6]. *Let f be a non-constant entire function and a, b be two distinct finite complex values and . If $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$ and $\overline{E}(b;f)=\overline{E}(b;f^{(1)})$, then $f\equiv f^{(1)}$.*

The uniqueness of an entire function sharing a nonzero finite value with its first two derivatives was considered by G. Jank, E. Mues and L. Volkmann [2] in 1986. The following is their result.

Theorem C. [2]. *Let f be a nonconstant entire function and a be a nonzero finite value. If $\overline{E}(a;f)=\overline{E}(a;f^{(1)})\subset\overline{E}(a;f^{(2)})$, then $f\equiv f^{(1)}$.*

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Considering $f = e^{\omega z} + \omega - 1$ and $a = \omega$, where ω is a $(k-1)$th imaginary root of unity and $k(\geq 3)$ is an integer, H. Zhong [9] pointed out that in Theorem A one cannot replace the second derivative by any higher order derivative. Under this context H. Zhong [9] proved the following theorem.

Theorem D. [9]. *Let f be a non-constant entire function and a be a nonzero finite number. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n(\geq 1)$, then $f \equiv f^{(n)}$.*

I. Lahiri and I. Kaish [3] improved Theorem D by considering a shared polynomial. They proved the following theorem.

Theorem E. [3]. *Let f be a non-constant entire function and $a = a(z)(\not\equiv 0)$ be a polynomial with $deg(a) \neq deg(f)$. Suppose that $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)})\backslash\{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$, where Δ denotes the symmetric difference of sets and $n(\geq 1)$ is an integer.*
If
(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,
(ii) $N_B(r, a; f^{(1)}) = S(r, f)$, and
(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,
then either $f = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Throughout the paper we denote by L a nonconstant linear differential polynomial in f of the form

$$L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}, \tag{1.1}$$

where $a_1, a_2, \ldots, a_n(\neq 0)$ are constants.
In 1999 P. Li [5] considered linear differential polynomials and proved the following result.

Theorem F. [5]. *Let f be a nonconstant entire function and L be defined by (1.1) and $a(\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f \equiv f^{(1)} \equiv L$.*

Considering a shared linear polynomial, we establish the following theorem which is our main result in the paper.

Theorem 1.1. *Let f be a nonconstant entire function, $a = a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constants and $k(\geq 1)$ be an integer. Further suppose that L defined by (1.1) be such that $L^{(k+1)}$ is nonconstant and*
(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$, where $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$;
(ii) $N_B(r, a; f^{(1)}) = S(r, f)$, where $B = \overline{E}(a; f^{(1)})\backslash\{\overline{E}(a; L^{(k)}) \cap \overline{E}(a; L^{(k+1)})\}$;
(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity.
Then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Putting $A = B = \Phi$, we obtain the following corollary.

Corollary 1.1. *Let f be a nonconstant entire function, $a = a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constants and $k(\geq 1)$ be an integer. Further suppose that L defined by (1.1) be such that $L^{(k+1)}$ is nonconstant and*
$E(a; f) = E(a; f^{(1)})$, $\overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; L^{(k)}) \cap \overline{E}(a; L^{(k+1)})\}$.
Then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. [3]. *Let f be transcendental entire function of finite order and $a = a(z)(\not\equiv 0)$, be a polynomial and $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$.*
If
(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,
(ii) each common zero of $f - a$ and $f^{(1)} - a$ have the same multiplicity,
then $m(r, a; f) = m(r, \frac{1}{f-a}) = S(r, f)$.

Lemma 2.2. *Let f be a transcendental entire function, $a = a(z) = \alpha z + \beta$, where $\alpha(\neq 0), \beta$ are constant and L defined by (1.1) be such that $L^{(k+1)}$ is nonconstant. Further suppose that*
$h = \frac{(a-a^{(1)})L^{(k)}-a(f^{(1)}-a^{(1)})}{f-a}$
and $A = \overline{E}(a;f)\backslash\overline{E}(a;f^{(1)})$
$B = \overline{E}(a;f^{(1)})\backslash\{\overline{E}(a;L^{(k)})\cap\overline{E}(a;L^{(k+1)})\}$, *$k(\geq 1)$ be an integer.*
If

(i) $N_A(r,a;f) + N_B(r,a;f^{(1)}) = S(r,f)$,
(ii) each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity,
(iii) h is transcendental entire or meromorphic function,

then $m(r,a;f^{(1)}) = m(r,\frac{1}{f^{(1)}-a}) = S(r,f)$.

Proof. From the hypothesis we get

$$\begin{aligned} N(r,h) &\leq N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f) + S(r,f) \\ &= N_{(2}(r,a;f) + S(r,f), \end{aligned} \tag{2.1}$$

where $N_{(2}(r,a;f)$ is the counting function of multiple zeros of $f-a$.

Let z_0 is a common zero of $f-a$ and $f^{(1)}-a$ with multiplicity $q(\geq 2)$, then z_0 is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q-1$.

So

$$N_{(2}(r,a;f) \leq 2N(r,0;a-a^{(1)}) + N_A(r,a;f) = S(r,f),$$

Therefore from (2.1) we get $N(r,h) = S(r,f)$.

Since $m(r,h) = S(r,f)$, we have $T(r,h) = S(r,f)$.

Now by a simple calculation we get
$f = a + \frac{1}{h}\{(a-a^{(1)})(L^{(k)}-a) - a(f^{(1)}-a)\}$.

Differentiating we obtain
$f^{(1)} = a^{(1)} + (\frac{1}{h})^{(1)}\{(a-a^{(1)})(L^{(k)}-a) - a(f^{(1)}-a)\}$
$+ (\frac{1}{h})\{a^{(1)}(L^{(k)}-a) + (a-a^{(1)})(L^{(k+1)}-a^{(1)}) - a^{(1)}(f^{(1)}-a) - a(f^{(2)}-a^{(1)})\}$.

That is,

$$\begin{aligned} (f^{(1)}-a)\left\{1 + (\frac{a}{h})^{(1)}\right\} &= (a^{(1)}-a) + \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(L^{(k)}-a) \\ &\quad + \left(\frac{a-a^{(1)}}{h}\right)(L^{(k+1)}-a^{(1)}) - \left(\frac{a}{h}\right)(f^{(2)}-a^{(1)}). \end{aligned} \tag{2.2}$$

Case 1:

First we suppose $k=1$. Then from (2.2) we get

$$\begin{aligned} (f^{(1)}-a)\left\{1 + (\frac{a}{h})^{(1)}\right\} &= (a^{(1)}-a) + \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(L^{(1)}-a) + \left(\frac{a-a^{(1)}}{h}\right)(L^{(2)}-a^{(1)}) \\ &\quad - \left(\frac{a}{h}\right)(f^{(2)}-a^{(1)}) \\ &= (a^{(1)}-a) + \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(a_1 a^{(1)}-a) + \left(\frac{a-a^{(1)}}{h}\right)^{(1)}(L^{(1)}-a_1 a^{(1)}) \\ &\quad + \left(\frac{a-a^{(1)}}{h}\right)L^{(2)} - \left(\frac{a-a^{(1)}}{h}\right)a^{(1)} - \left(\frac{a}{h}\right)(f^{(2)}-a^{(1)}). \end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{f^{(1)}-a} &= \frac{\zeta}{\eta}-\frac{1}{\eta}\left(\frac{a-a^{(1)}}{h}\right)^{(1)}\frac{L^{(1)}-a_1a^{(1)}}{f^{(1)}-a}-\frac{a-a^{(1)}}{h\eta}\cdot\frac{L^{(2)}}{f^{(1)}-a}\\ &\quad+\frac{a}{h\eta}\cdot\frac{f^{(2)}-a^{(1)}}{f^{(1)}-a},\end{aligned}\tag{2.3}$$

where $\zeta = 1+\left(\frac{a}{h}\right)^{(1)}$ and $\eta = (a^{(1)}-a)+\left(\frac{a-a^{(1)}}{h}\right)^{(1)}(a_1a^{(1)}-a)-\left(\frac{a-a^{(1)}}{h}\right)a^{(1)}$.

We now verify that $\zeta \not\equiv 0$ and $\eta \not\equiv 0$. If $\zeta \equiv 0$, then $1+(\frac{a}{h})^{(1)} \equiv 0$. Integrating we get $h = \frac{a}{d-z}$, where d is a constant, and this implies a contradiction as h is transcendental.

If $\eta \equiv 0$, then

$(a^{(1)}-a)+\left(\frac{a-a^{(1)}}{h}\right)^{(1)}(a_1a^{(1)}-a)-\left(\frac{a-a^{(1)}}{h}\right)a^{(1)} \equiv 0$

Integrating we get

$h = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials of degree 2, which is a contradiction as h is transcendental.

Since clearly $T(r,\zeta)+T(r,\eta) = S(r,f)$, from (2.3) we get $m(r,a;f^{(1)}) = m(r,\frac{1}{f^{(1)}-a}) = S(r,f)$.

Case 2:

Next we suppose $k>1$. Then from (2.2) we get

$$\begin{aligned}(f^{(1)}-a)\left\{1+(\frac{a}{h})^{(1)}\right\} &= (a^{(1)}-a)-\left(\frac{a(a-a^{(1)})}{h}\right)^{(1)}+\left(\frac{a-a^{(1)}}{h}\right)^{(1)}L^{(k)}\\ &\quad+\left(\frac{a-a^{(1)}}{h}\right)L^{(k+1)}-\left(\frac{a}{h}\right)(f^{(2)}-a^{(1)})\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{f^{(1)}-a} &= \frac{\zeta}{\eta_1}-\frac{1}{\eta_1}\left(\frac{a-a^{(1)}}{h}\right)^{(1)}\frac{L^{(k)}}{f^{(1)}-a}-\frac{a-a^{(1)}}{h\eta_1}\cdot\frac{L^{(k+1)}}{f^{(1)}-a}\\ &\quad+\frac{a}{h\eta_1}\cdot\frac{f^{(2)}-a^{(1)}}{f^{(1)}-a},\end{aligned}\tag{2.4}$$

where $\zeta = 1+\left(\frac{a}{h}\right)^{(1)}$ and $\eta_1 = (a^{(1)}-a)-\left(\frac{a(a-a^{(1)})}{h}\right)^{(1)}$.

In case 1 we see that $\zeta \not\equiv 0$. We now verify that $\eta_1 \not\equiv 0$. If $\eta_1 \equiv 0$, then

$(a^{(1)}-a)-\left(\frac{a(a-a^{(1)})}{h}\right)^{(1)} \equiv 0.$

Integrating we get, $h = \frac{P_1(z)}{Q_1(z)}$, where $P_1(z)$ and $Q_1(z)$ are polynomials of degree 2, which is a contradiction as h is transcendental.

Since clearly $T(r,\zeta)+T(r,\eta_1) = S(r,f)$, from (2.4) we get $m(r,a;f^{(1)}) = m(r,\frac{1}{f^{(1)}-a}) = S(r,f)$.

This proves the lemma. □

Lemma 2.3. *Let f be a transcendental entire function, $a = a(z) = \alpha z+\beta$, where $\alpha(\neq 0),\beta$ are constants and $k(\geq 1)$ be an integer. Further suppose that L defined by (1.1) be such that $L^{(k+1)}$ is nonconstant and*

(i) $N_A(r,a;f)+N_A(r,a;f^{(1)}) = S(r,f)$, where $A = \overline{E}(a;f)\Delta\overline{E}(a;f^{(1)})$;
(ii) $N_B(r,a;f^{(1)}) = S(r,f)$, where $B = \overline{E}(a;f^{(1)})\backslash\{\overline{E}(a;L^{(k)})\cap\overline{E}(a;L^{(k+1)})\}$;
(iii) each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity;
(iv) $m(r,a;f) = S(r,f)$.

Then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Proof. Let

$$\phi = \frac{f^{(1)}-a}{f-a}.\tag{2.5}$$

By the hypotheses, $N(r,\phi) \leq N_A(r,a;f) + S(r,f) = S(r,f)$ and

$$m(r,\phi) = m\left(r, \frac{f^{(1)} - a^{(1)} + a^{(1)} - a}{f - a}\right) \leq m(r,a;f) + S(r,f) = S(r,f).$$

Therefore $T(r,\phi) = S(r,f)$. Now from (2.5) we get

$$f^{(1)} = \mu_1 + \nu_1 f, \tag{2.6}$$

where $\mu_1 = a(1-\phi)$ and $\nu_1 = \phi$.

Differentiating (2.6) and using it again, we get

$$f^{(2)} = \mu_2 + \nu_2 f, \tag{2.7}$$

where $\mu_2 = \mu_1^{(1)} + \nu_1\mu_1$ and $\nu_2 = \nu_1^{(1)} + \nu_1\nu_1$.

Differentiating (2.7) and using (2.6) we get

$$f^{(3)} = \mu_3 + \nu_3 f,$$

where $\mu_3 = \mu_2^{(1)} + \mu_1\nu_2$ and $\nu_3 = \nu_2^{(1)} + \nu_2\nu_1$.

In general, we obtain

$$f^{(j)} = \mu_j + \nu_j f, \tag{2.8}$$

where $\mu_{j+1} = \mu_j^{(1)} + \mu_1\nu_j$ and $\nu_{j+1} = \nu_j^{(1)} + \nu_1\nu_j$ for $j = 1, 2, 3, \ldots$.

Clearly $T(r,\mu_j) + T(r,\nu_j) = S(r,f)$ for $j = 1, 2, 3, \ldots$. Let z_0 be a pole of ϕ with multiplicity $p(\geq 2)$. Then z_0 is a pole of ν_2 with multiplicity $\max\{2p, p+1\} = 2p$ and is a pole of ν_3 with multiplicity $\max\{3p, 2p+1\} = 3p$. In general, z_0 is a pole of ν_{j+1} with multiplicity $\max\{(j+1)p, jp+1\} = (j+1)p$.

Now

$$L^{(k)} = \sum_{j=1}^{n} a_j f^{(j+k)} = \sum_{j=1}^{n} a_j \mu_{j+k} + \left(\sum_{j=1}^{n} a_j \nu_{j+k}\right) f = \xi + \chi f, \text{say}. \tag{2.9}$$

Clearly $T(r,\xi) + T(r,\chi) = S(r,f)$. Differentiating (2.9) we obtain

$$L^{(k+1)} = \xi^{(1)} + \chi^{(1)} f + \chi f^{(1)}. \tag{2.10}$$

Let $E^* = \overline{E}(a;f) \cap \overline{E}(a;f^{(1)}) \cap \overline{E}(a;L^{(k)}) \cap \overline{E}(a;L^{(k+1)})$. We note that $E^* \neq \emptyset$ because otherwise $N(r,a;f) = S(r,f)$, a contradiction. If $z_1 \in E^*$, then from (2.9) and (2.10) we get

$$\xi(z_1) + \chi(z_1)a(z_1) = a(z_1) \tag{2.11}$$

and

$$\xi^{(1)}(z_1) + \chi^{(1)}(z_1)a(z_1) + \chi(z_1)a(z_1) = a(z_1). \tag{2.12}$$

We put $\gamma = \xi + \chi a - a$ and $\delta = \xi^{(1)} + \chi^{(1)}a + \chi a - a$. Then $T(r,\gamma) + T(r,\delta) = S(r,f)$. If $\gamma \not\equiv 0$, then from (2.11) we get $N_{E^*}(r,a;f) = \overline{N}_{E^*}(r,a;f) \leq N(r,0;\gamma) = S(r,f)$. Now

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N_B(r,a;f^{(1)}) + N_{E^*}(r,a;f) + S(r,f) \\ &= S(r,f) \end{aligned}$$

and by hypothesis $T(r,f) = S(r,f)$, a contradiction. So $\gamma \equiv 0$. Similarly we can show that $\delta \equiv 0$. This implies that $\xi \equiv 0$ and $\chi \equiv 1$. From (2.9) we get $f \equiv L^{(k)}$.

Also $\sum_{j=1}^{n} a_j \nu_{j+k} \equiv 1$ and $a_n \neq 0$ imply that ϕ has no pole. For, otherwise the left hand side would have some pole while the right hand side is a constant, a contradiction.

Further from $\sum_{j=1}^{n} a_j \nu_{j+k} \equiv 1$ we get

$$a_n \phi^{n+k} + P[\phi] \equiv 0, \tag{2.13}$$

where $P[\phi]$ is a differential polynomial in ϕ with degree not exceeding $n+k-1$.

If ϕ is a transcendental entire function, then by Clunie's lemma we have $m(r,\phi) = S(r,\phi)$, a contradiction. If ϕ is a nonconstant polynomial, then the left hand side of (2.13) is a nonconstant polynomial, which is impossible. Therefore ϕ is a constant and we get

$f^{(1)}-a=\phi(f-a)$, i.e., $f^{(1)}-\phi f=a(1-\phi)$, i.e., $\dfrac{df}{dz}-\phi f=(\alpha z+\beta)(1-\phi)$.

This implies $\dfrac{d}{dz}(e^{-\phi z}f)=e^{-\phi z}(\alpha z+\beta)(1-\phi)$ and so on integration we obtain

$$e^{-\phi z}f=\frac{a(\phi-1)}{\phi}e^{-\phi z}(a+\frac{\alpha}{\phi})+c,$$

i.e.,

$$f=\frac{\phi-1}{\phi}+\lambda e^{\phi z},$$

where $\lambda(\neq 0)$ is a constant.

Therefore

$$\begin{aligned} L^{(k)} &= \sum_{j=1}^{n}a_jf^{(j+k)}=\left(\sum_{j=1}^{n}a_j\phi^{j+k}\right)\lambda e^{\phi z} \\ &= \left(\sum_{j=1}^{n}a_j\nu_{j+k}\right)\lambda e^{\phi z}=\lambda e^{\phi z}=\frac{f^{(1)}}{\phi}-\frac{\phi-1}{\phi^2}\left(a+\frac{\alpha}{\phi}\right). \end{aligned} \tag{2.14}$$

Since $E^*\neq\emptyset$, we have $f(z_2)=f^{(1)}(z_2)=L^{(k)}(z_2)=L^{(k+1)}(z_2)=a(z_2)$ for some $z_2\in E^*$.

Then from (2.14) we get

$a(z_2)=\frac{a(z_2)}{\phi}-\frac{\phi-1}{\phi^2}\left(a(z_2)+\frac{\alpha}{\phi}\right)$. By simple calculation we get $(\phi-1)\{(\phi+1)a(z_2+\alpha\}=0$

If $\phi\neq 1$, then $\{(\phi+1)a(z_2)+\alpha\}=0$ and by the hypothesis we get $N_{E^*}(r,a;f)=\overline{N}_{E^*}(r,a;f)\leq N(r,0;\{(\phi+1)a+\alpha\})=S(r,f)$. Now

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f)+N_B(r,a;f^{(1)})+N_{E^*}(r,a;f)+S(r,f) \\ &= S(r,f) \end{aligned}$$

and by hypothesis $T(r,f)=S(r,f)$, a contradiction.

Therefore $\phi=1$ and $L^{(k)}\equiv f\equiv\lambda e^z$.

Now we have

$$\lambda e^z=L^{(k)}=\sum_{j=1}^{n}a_jf^{(j+k)}=\left(\sum_{j=1}^{n}a_j\right)\lambda e^z \quad \text{and so} \quad \sum_{j=1}^{n}a_j=1.$$

Therefore $L=\sum_{j=1}^{n}a_jf^{(j)}=\left(\sum_{j=1}^{n}a_j\right)\lambda e^z=\lambda e^z$. This proves the lemma. □

Lemma 2.4. $\{$*p.58* [7]$\}$. *Each solution of the differential equation*

$$a_nf^{(n)}+a_{n-1}f^{(n-1)}+\cdots+a_0f=0,$$

where $a_0(\not\equiv 0),a_1,\cdots,a_n(\not\equiv 0)$ *are polynomials, is an entire function of finite order.*

Lemma 2.5. $\{$ *p.47* [1] $\}$. *Let* f *be a nonconstant meromorphic function and* a_1*,* a_2*,* a_3 *be three distinct meromorphic functions satisfying* $T(r,a_\nu)=S(r,f)$ *for* $\nu=1,2,3$.

Then $T(r,f)\leq\overline{N}(r,0;f-a_1)+\overline{N}(r,0;f-a_2)+\overline{N}(r,0;f-a_3)+S(r,f)$.

3. Proof of The Theorem

First we claim that f is a transcendental entire function. If f is a polynomial, then $T(r,f)=O(\log r)$ and $N_A(r,a;f)+N_A(r,a;f^{(1)})=O(\log r)$. Then from the hypothesis we get $O(\log r)=O(\log T(r,f))=S(r,f)$, which implies $T(r,f)=S(r,f)$, a contradiction. Therefore $A=\emptyset$. Similarly, $N_B(r,a;f^{(1)})=S(r,f)$ implies $B=\emptyset$.

Therefore $E(a;f)=E(a;f^{(1)})$ and $\overline{E}(a;f^{(1)})\subset\overline{E}(a,L^{(k)})\cap\overline{E}(a;L^{(k+1)})$.

First we suppose degree of f be 1 and we consider $f(z)=Az+B$, where $A(\neq 0),B$ are constants. Then $f^{(1)}=A$ and $L^{(k)}\equiv L^{(k+1)}\equiv 0$. Now $\frac{A-\beta}{\alpha}$ is the only zero of $f^{(1)}-a$ and $-\frac{\beta}{\alpha}$ is the only zero of $L^{(k)}-a$. So $\overline{E}(a;f^{(1)})\subset\overline{E}(a,L^{(k)})$ implies that $\frac{A-\beta}{\alpha}=-\frac{\beta}{\alpha}$ and so $A=0$, a contradiction.

Next we suppose that the degree of f be greater than 1. Then $deg(f-a) > deg(f^{(1)}-a)$. Since each common zero of $f-a$ and $f^{(1)}-a$ has the same multiplicity, this contradicts the fact that $E(a;f) = E(a;f^{(1)})$.

Therefore our claim 'f is a transcendental entire function' is established.

We note that a common zero of $f-a$ and $f^{(1)}-a$ of multiplicity $q(\geq 2)$ is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q-1(\geq 1)$. Therefore

$$N_{(2}(r,a;f^{(1)}|f=a) \leq 2N(r,0;a-a^{(1)}) = S(r,f),$$

where $N_{(2}(r,a;f^{(1)}|f=a)$ denotes the counting function (counted with multiplicities) of those multiple zeros of $f^{(1)}-a$, which are also zeros of $f-a$.

Now

$$\begin{aligned} N_{(2}(r,a;f^{(1)}) &\leq N_A(r,a;f^{(1)}) + N_B(r,a;f^{(1)}) + N_{(2}(r,a;f^{(1)}|f=a) + S(r,f) \\ &= N_{(2}(r,a;f^{(1)}|f=a) + S(r,f) \\ &= S(r,f), \end{aligned} \tag{3.1}$$

where $N_{(2}(r,a;f^{(1)})$ denotes the counting function (counted with multiplicities) of multiple zeros of $f^{(1)}-a$.

Case 1: Let $L^{(k)} \equiv L^{(k+1)} \equiv f^{(1)}$.

Then $L^{(k)} = L^{(k+1)} = f^{(1)} = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Therefore $f = \lambda e^z + t$, where t is a constant.

By Lemma 2.5 we get

$$\begin{aligned} T(r,\lambda e^z) &\leq \overline{N}(r,0;\lambda e^z) + \overline{N}(r,\infty;\lambda e^z) + \overline{N}(r,a-t;\lambda e^z) + S(r,\lambda e^z) \\ &= \overline{N}(r,a;f) + S(r,\lambda e^z), \end{aligned}$$

which implies $\overline{N}(r,a;f) \neq S(r,f)$. Again since $N_A(r,a;f) + N_B(r,a;f^{(1)}) = S(r,f)$, we get $\overline{E}(a;f) \cap \overline{E}(a;f^{(1)}) \neq \emptyset$, otherwise $\overline{N}(r,a;f) \neq S(r,f)$. This implies $t=0$ and so $f = L^{(k)}$.

Now $m(r,a;f) \leq m(r,\frac{a}{f-a}+1) + S(r,f) = m(r,\frac{f}{f-a}) + S(r,f) = m(r,\frac{L^{(k)}}{f-a}) + S(r,f) = S(r,f)$

Therefore by Lemma 2.3 we get $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Case 2: Let $L^{(k+1)} \not\equiv f^{(1)}$. Then using (3.1) we get by the hypothesis

$$\begin{aligned} N(r,a;f^{(1)}) &\leq N_B(r,a;f^{(1)}) + N(r,\frac{a}{a-\alpha};\frac{L^{(k+1)}}{f^{(1)}-\alpha}) + S(r,f) \\ &\leq T(r,\frac{L^{(k+1)}}{f^{(1)}-\alpha}) + S(r,f) \\ &= N(r,\frac{L^{(k+1)}}{f^{(1)}-\alpha}) + S(r,f) \\ &\leq N(r,\alpha;f^{(1)}) + S(r,f). \end{aligned} \tag{3.2}$$

Again

$$\begin{aligned} m(r,a;f) &\leq m(r,\frac{f^{(1)}-\alpha}{f-a}\cdot\frac{1}{f^{(1)}-\alpha}) \\ &\leq m(r,\alpha;f^{(1)}) + S(r,f) \\ &= T(r,f^{(1)}) - N(r,\alpha;f^{(1)}) + S(r,f) \\ &= m(r,f^{(1)}) - N(r,\alpha;f^{(1)}) + S(r,f) \\ &\leq m(r,f) - N(r,\alpha;f^{(1)}) + S(r,f) \\ &= T(r,f) - N(r,\alpha;f^{(1)}) + S(r,f) \end{aligned}$$

i.e., $N(r,\alpha;f^{(1)}) \leq N(r,a;f) + S(r,f)$.

So from (3.2) we get

$$N(r,a;f^{(1)}) \leq N(r,a;f) + S(r,f). \tag{3.3}$$

Again

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N(r,a;f^{(1)} \mid f=a) \\ &\leq N(r,a;f^{(1)}) + S(r,f). \end{aligned} \tag{3.4}$$

Therefore from (3.3) and (3.4) we get

$$N(r,a;f^{(1)}) = N(r,a;f) + S(r,f). \tag{3.5}$$

Let h, defined as in Lemma 2.2, be transcendental. Then

$$\begin{aligned} T(r,f) &= m(r,f) = m(r,\frac{1}{h}\{(a-a^{(1)})(L^{(k)}-a) - a(f^{(1)}-a)\}) + S(r,f) \\ &\leq m(r,\frac{1}{h}\{(a-a^{(1)})L^{(k)} - af^{(1)}\}) + S(r,f) \\ &\leq m(r,f^{(1)}) + S(r,f) \\ &= T(r,f^{(1)}) + S(r,f) \\ &= m(r,f^{(1)}) + S(r,f) \\ &\leq m(r,f) + S(r,f) \\ &= T(r,f) + S(r,f). \end{aligned}$$

Therefore

$$T(r,f^{(1)}) = T(r,f) + S(r,f). \tag{3.6}$$

Again by Lemma 2.2 we get $m(r,a;f^{(1)}) = S(r,f)$.
Then from (3.5) and (3.6) we get $m(r,a;f) = S(r,f)$.
Next we suppose that h is rational. Then by Lemma 2.4 we see that f is of finite order.
So by the hypothesis and Lemma 2.1 we get $m(r,a;f) = S(r,f)$.
Therefore by Lemma 2.3 we get $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.
Case 3: Finally let $L^{(k+1)} \not\equiv L^{(k)}$.
Then by the hypothesis and (3.1) we get

$$\begin{aligned} N(r,a;f^{(1)}) &\leq N_B(r,a;f^{(1)}) + N(r,1;\frac{L^{(k+1)}}{L^{(k)}}) + S(r,f) \\ &\leq T(r,\frac{L^{(k+1)}}{L^{(k)}}) + S(r,f) \\ &= N(r,\frac{L^{(k+1)}}{L^{(k)}}) + S(r,f) \\ &= \overline{N}(r,0;L^{(k)}) + S(r,f). \end{aligned} \tag{3.7}$$

Again

$$\begin{aligned} m(r,a;f) &= m(r,\frac{L^{(k)}}{f-a}\cdot\frac{1}{L^{(k)}}) \\ &\leq m(r,0;L^{(k)}) + S(r,f) \\ &= T(r,L^{(k)}) - N(r,0;L^{(k)}) + S(r,f) \\ &= m(r,L^{(k)}) - N(r,0;L^{(k)}) + S(r,f) \\ &\leq m(r,\frac{L^{(k)}}{f}) + m(r,f) - N(r,0;L^{(k)}) + S(r,f) \\ &= m(r,f) - N(r,0;L^{(k)}) + S(r,f) \\ &= T(r,f) - N(r,0;L^{(k)}) + S(r,f) \end{aligned}$$

and so
$N(r,0;L^{(k)}) \leq N(r,a;f) + S(r,f)$. Now by (3.7) we get

$$N(r,a;f^{(1)}) \leq N(r,a;f) + S(r,f). \tag{3.8}$$

Also

$$\begin{aligned} N(r,a;f) &\leq N_A(r,a;f) + N(r,a;f^{(1)} \mid f=a) \\ &\leq N(r,a;f^{(1)}) + S(r,f). \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we get $N(r,a;f^{(1)}) = N(r,a;f) + S(r,f)$, which is (3.5).
Now as in Case 2, by Lemma 2.1, Lemma 2.2, Lemma 2.4 and (3.5) we obtain $m(r,a;f) = S(r,a;f)$.
Then by Lemma 2.3 we get $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.
This proves the theorem.

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