Overview of Accretive Operator

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ABSTRACT: *This paper is a survey on the results of Accretive Operators, which are the generalization of Monotone Operators in Banach space. We have also discussed different classes of Accretive Operators and their application in solving various problems.*

Keywords: *Accretive Operator, Duality mapping, Monotone Operator*.

2020 AMS subject classification: 47H06.

I. Introduction

Functional Analysis is an area of Mathematics that has grown over the past few decades, influenced by problems existing in Physics, Mechanics, Operation Research and Economics. It can be categorized into two parts - Linear and Nonlinear. Most problems of the physical world are nonlinear and these problems are modeled in the form of mathematical equations. Mathematical models for a large number of problems in science lead to equations $Fx = y$ in infinite dimensional spaces[15]. In particular, all kinds of differential equations, integral equations, integrodifferential equations, etc. can be formulated this way on usually infinite dimensional spaces of functions. The next task is to find the solutions, concerned about the uniqueness of the solution and also the distribution of such solutions in the domain of F.

The Operator plays a key role in these mathematical models. "Accretive Operator" is one such operator introduced independently by Browder [5] and Kato [19]. Of course, the reason was its capacity in dealing with certain nonlinear models.

II. Generalization of Monotone Operator to Accretive Operator

2.1 Monotone Operator in Hilbert Space

The concept of the monotone operator was first introduced by Minty [24] in 1962, Minty defined the monotone operators as:

Definition 2.1 *[24]*A mapping $F: D \subseteq H \rightarrow H$ which satisfies

$$(Fx - Fy, x - y) \geq 0 \quad \forall x, y \in D, \tag{1}$$

where H is a Hilbert space with inner product (. , .) is called monotone.

Examples of monotone mapping [15] are:

Example 2.1 Let $F: R \rightarrow R$ be a monotone increasing function. Then F is a monotone operator.

Example 2.2 In the case when $H = R^n$ and $F(x) = Ax$ is linear, then condition (1) just means $(Ax, x) \geq 0$ on R^n i.e., monotonicity is the same as positive semidefiniteness of the matrix A.

2.2 Monotone Operator in Banach Space

Among all infinite-dimensional Banach spaces, Hilbert spaces have the nicest geometric properties. The following two identities

$$||x + y||^2 = ||x||^2 + 2 < x, y > + ||y||^2,$$

$$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 - \lambda(1 - \lambda)||x - y||^2 + (1 - \lambda)||y||^2$$

,

which hold for all $x, y \in H$, are some of the geometric properties that characterize innerproduct spaces and also make certain problems posed in Hilbert spaces more manageable than those in general Banach spaces.

M. Hazewinkel observed, that "Many and

probably most, mathematical objects and models do not naturally live in Hilbert spaces"[12]. There are many interesting initial value problems in the theory of partial differential equations whose natural setting is not a Hilbert space, but rather a Banach space. Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the above identities were developed using the duality map which has become one of the most important tools in nonlinear functional analysis. The developments include the work of Bynum[11], Reich[28], Xu[30], Xu and Roach[31] and many others.

The concept of monotonicity can be generalized from Hilbert spaces to more general Banach spaces. Let E be a real Banach space and E^* its dual space, then $F: E \to E^*$ is said to be monotone if

$$(Fx_1 - Fx_2, x_1 - x_2) \geq 0, \text{ for all } x_1, x_2 \in D(F),$$

where $(x, x*)$ denotes the value of $x^* \in E^*$ at $x \in E$. When $E = H$ is Hilbert space, then $E^* = E$ and (x, x^*) denotes the inner product in H.

2.3 Duality mapping

Let E be a real Banach space, E^* its dual space with the duality pairing between w in E^* and x in E, denoted by (x, w). Beurling and Livingston [1] introduced the concept of duality mapping $J: E \to 2^{E^*}$ as

$$J(x) = \{w: w \in E^*, ||w|| = ||x||; (x, w) = ||x|| ||w||\},$$

for each x in E.

For any Banach space E and any element x of E, $J(x)$ is a nonempty closed convex subset of the sphere of radius $||x||$ about zero in E^*. If E^* is strictly convex, J is a single-valued mapping of E into E^* and is continuous from the strong topology of E to the $weak^*$ topology of E^*.

The single valued duality mapping [5] is defined as $J: E \to E^*$ such that for each x in E, $(Jx, x) = ||x||.||Jx||$ and $||x|| = ||Jx||$.

In the monotonicity problems, the duality mapping takes the place of the identity in Hilbert spaces and so the duality pair is replaced by the respective inner product. Moreover, it is the key technique in the study of evolution equations related to monotone and accretive operators.

2.4 J-Monotone operator

Browder and Figueiredo [9] introduced the concept of J-monotonicity using the concept of duality mapping.

A mapping $F: D(F) \subseteq E \to E$ is said to be J-monotone if

$$(Fx_1 - Fx_2, J(x_1 - x_2)) \geq 0 \text{ , for all } x_1, x_2 \in D(F),$$

where $J: E \to E^*$ is a given duality mapping. This class of operators has also been studied by Browder [3, 4].

When the duality mapping J is multi-valued, then the definition of J-monotone operators coincides with an important class of operators known as "Accretive operators".

2.5 Accretive Operator

The concept of the Accretive operator is given by F. E. Browder [5] and T. Kato [19] in 1967.

Browder [5] defined the Accretive operators as

Definition 2.2 Let A be a nonlinear mapping with domain D(A) and range R(A) in real Banach space E, then A is said to be accretive if for each pair x and y in D(A),

$$(A(x) - A(y), w) \geq 0, \forall w \in J(x - y), \quad (2)$$

where $J: E \to 2^{E^*}$ is a given duality mapping.

An element $x^* \in E$ is said to be zero of the accretive operator A, if $Ax^* = 0$ and is respectively, said to be the solution of the accretive operator equation $Ax = 0$.

When E^* is strictly convex (i.e. J is single-valued), the nonlinear accretive operators from E to E coincide with the J-monotone operators. Whereas, if E is a Hilbert space, then J = I and the accretive operators reduce to the class of monotone operators.

Kato [19] defined the accretive operators as

Definition 2.3 An operator A with domain D(A) and range R(A) in an arbitrary Banach space E is said to be accretive if

$$||x - y + \alpha(Ax - Ay)|| \geq ||x - y|| \quad (3)$$

for every $x, y \in D(A)$ and $\alpha > 0$.

Kato [19] showed that the accretivity thus defined can also be expressed in terms of the duality mapping J from E to E^* (in general, a multi-valued operator). For this, Kato proved a lemma generally known as "Kato's lemma", which states that,

Lemma 2.1 ***<u>Kato's Lemma</u>***:-

Let $x, y \in E$. Then $||x|| \leq ||x + \alpha y||$, for every $\alpha > 0$ if and only if, there is $j \in Jx$ such that $Re(y, j) \geq 0$.

Using this lemma, Kato showed that (3) is equivalent to the following definition of accretive operator:-

For each $x, y \in D(A)$, there exist $j \in J(x - y)$ such that

$$Re(Ax - Ay, j) \geq 0. \qquad (4)$$

The notion of accretive operators in (2) introduced by Browder [5] in a Banach space E, is almost identical to that of accretive operators defined in (4) by Kato [19]. There is a slight difference that Browder requires $Re(Ax - Ay, j) \geq 0$, for every $j \in J(x - y)$, whereas Kato requires it only for some $j \in J(x - y)$.

These two definitions coincide if J is single-valued. Example of an Accretive Operator follows as:-

Example 2.3[13]Let R=$(-\infty, \infty)$ with usual norm and $A: [0,1] \to R$ be defined by $Ax = \frac{x}{2} - 1$. Then for $x, y \in [0,1]$

$$(Ax - Ay, j(x - y)) = |Ax - Ay||x - y|$$
$$= \frac{1}{2}|x - y|^2 \geq 0$$

Hence, A is accretive.

III. Different classes of Accretive Operators

3.1 Maximal Accretive Operator

Definition 3.1A mapping A is said to be **Maximal accretive**[14] if it is accretive and the inclusion $A \subseteq B$, with B accretive, implies $A = B$. (i.e. A is said to be maximal accretive if it is not properly contained in another accretive mapping.)

3.2 m-Accretive Operator

Definition 3.2 Let A be an accretive operator with domain D(A) and range R(A) in E, where E is a Banach space with dual E*. Then A is said to be **m-Accretive**[19] if the operator (I+rA) is surjective (i.e. R(I+rA)=E) for all $r > 0$, where I is the identity operator on E.

Browder [6] states that if $A: E \to E$ is locally Lipschitzian and accretive, then A is m-accretive. In 1970, Martin [23] extended this result by proving that if $A: E \to E$ is continuous and accretive, then A is m-accretive.

An example of a continuous accretive and hence m-accretive operator [21]:-

Example 3.1*[21] Let $E = (-\infty, \infty)$ with the usual norm $||.||$ and $A: E \to E$ be defined by*

$$Ax = \begin{cases} x - 1; & x \in (-\infty, -1) \\ x - \sqrt{-x}; & x \in [-1, 0) \\ x + \sqrt{x}; & x \in [0,1] \\ x + 1; & x \in (1, +\infty) \end{cases}$$

Then A is continuous and R(A) is unbounded.

To prove that A is accretive, consider, in particular, the case if $x, y \in (-\infty, -1)$ and $r > 0$. Then

$$||x - y + r(Ax - Ay)|| = (1 + r)||x - y|| \geq ||x - y||,$$

which implies that A is accretive.

Remark In Banach spaces, m-accretiveness always implies maximal accretiveness, but the converse is not true[15]. In Hilbert spaces, both notions coincide.

3.3 Strongly Accretive Operator

Let E be an arbitrary Banach space with dual E^*. An operator A with domain D(A) and range R(A) in E is called **Strongly Accretive** [27] if there exist a constant $k > 0$ such that for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$Re(Ax - Ay, j(x - y)) \geq k(||x - y||)^2 \qquad (5)$$

without loss of generality, we may assume $k \in (0,1)$.

If k=0 in (5), then A is called Accretive (Browder[7] and Kato [19]).

3.4 ϕStrongly Accretive Operator

Let E be an arbitrary Banach space with dual E^*. An operator A with domain D(A) and range R(A) in E is called ϕ**Strongly Accretive** [27] if there exist a strictly increasing function $\phi [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(A)$, there exist $j(x - y) \in J(x - y)$ satisfying

$$Re(Ax - Ay, j(x - y)) \geq \phi(||x - y||)||x - y||$$

Remark: The class of strongly accretive operators is a proper subclass of the class of $\phi - strongly$ accretive operators. i.e. Every strongly accretive operator is ϕstrongly accretive with $\phi [0, \infty) \to [0, \infty)$ defined by ϕ(s)=ks but the converse doesn't hold [27].

Example 3.2 *[27]*Let E = R, the reals with the

usual norm and let $K = [0, \infty)$.
Define $A: K \to K$ by

$$Ax = x - \frac{x}{1+x}$$

Then A is ϕstrongly accretive with $\phi [0,\infty) \to [0,\infty)$, defined by $\phi(s) = \frac{s^2}{1+s}$.

In particular, if x = 1 and y = 2, then

$$(A(1) - A(2), j(1-2)) = |A(1) - A(2)||1-2| = |\frac{1}{2} - \frac{4}{3}||-1| = \frac{5}{6},$$

whereas,

$$\phi(||1-2||)||1-2|| = \phi(|1-2|)|1-2| = \phi(1).(1) = \frac{1}{2},$$

which implies that A is ϕstrongly accretive i.e.

$$(Ax - Ay, j(x-y)) \geq \phi(||x-y||)||x-y||.$$

However, given any $k \in (0,1)$, if we choose $x \in K$ such that $0 < x < \frac{k}{1-k}$ and y=0, then to show that A is Strongly Accretive, we must have,

$$< Ax - Ay, x - y > \geq k|x-y|^2, i.e. < Ax - 0, x - 0 > \geq k|x-0|^2, or |x - \frac{x}{1+x}||x| \geq k|x|^2, \frac{x}{1+x} \geq k, i.e. x \geq \frac{k}{1-k},$$

but since $x \leq \frac{k}{1-k}$,hence

$$< Ax - Ay, x - y > \leq k|x-y|^2,$$

so that A is not Strongly Accretive.

IV. Application of Accretive Operators in solving various problems

In the area of nonlinear analysis, the theory of Accretive Operators is an important and developing field, due to its application in various areas as:-

- Accretive Operators is firmly connected with equations of evolutions found in the heat, wave, Schrodinger and similar other equations [15, 19].
- The solution of a non-linear evolution equation which gives a novel approach to the X-ray tomography problem can be obtained with the help of Accretive Operators[16].
- Many problems in Operation Research and Mathematical Physics can be written as variational inequalities, equilibrium problems, or operator inclusions with Accretive Operators [29].
- The result of finding common zeros of a finite family of accretive operators is useful in solving convex feasibility problems. The Convex feasibility problem captures applications in various disciplines such as sensor networking, radiation therapy, treatment planning, computerized tomography, and image restoration[26].

V. Relation of accretive operators with other mappings and fundamental results

5.1 Accretive Operators are closely related to various classes of other mappings

Accretive Operator and Non-expansive mapping:-

The relation between non-expansive and accretive mappings creates a strong connection between the fixed point theory of nonexpansive mappings and the operator theory of accretive maps. These relations are as follows[7]:

(a) If U is a nonexpansive mapping then $A = I - U$ is an accretive mapping

(b) If $\{U(t), t \geq 0\}$ is a semigroup of (nonlinear) mappings of E into E with infinitesimal generator A, then all the mappings U(t) are nonexpansive if and only if $(-A)$ is accretive.

Accretive operator and Pseudo-contractive mapping:-

Pseudo-contractive mappings introduced by Browder and Petryshyn[10] are defined as:

If T is a mapping of D(T) in E, into E, then T is said to be pseudo-contractive if for each $x_1, x_2 \in D(T)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$(Tx_1 - Tx_2, j(x_1 - x_2)) \leq ||x - y||^2$$

The relation between Accretive Operator and Pseudo contractive mapping was given by Browder[7] in the following proposition:

Proposition 5.1 [7] Let E be a Banach space, T is a mapping with domain and range in E, $A = I - T$. Then T is pseudo-contractive if and only if A is accretive.

Accretive operator and Dissipative Operator:-
In the linear case, A is accretive if and only if $(-A)$ is dissipative in the sense of Lumer-Phillips [22].

5.2 Fundamental Results on Accretive operators

Browder [5] established several general existence theorems for solutions of nonlinear functional equations involving nonlinear accretive operators which significantly improve earlier results in this direction. Further, Browder[6] presented some new and sharper results on two related topics:

1. The existence theory of solutions for the initial value problem for nonlinear equations of evolution of the form

$$\frac{du}{dt} + A(t)u(t) = f(t, u(t)), (t \geq 0), \quad (6)$$

with the initial condition $u(0) = x_0$, where it is assumed that for each t in R^+, A(t) is an accretive operator such that D(A(t)) is independent of t and $R(A(t) + I) = E$, while f is a continuous, bounded mapping of $R^+ \times E$ into E.

2. The existence theory of solutions of the equation

$$A(u) = w \quad (7)$$

for an accretive operator A and an element w of E.

In the special case,

- when E is a Hilbert space and A(t) is linear, such results were obtained by Browder [2] and Kato [18],
- extensions for A(t) linear and more general Banach spaces E, were given in Murakami [25] and Browder [5],
- results for A(t) nonlinear were first obtained by komura [20] in Hilbert space and
- extended to more general Banach spaces by Kato [19] for the case in which $f = 0$.

The connection between the equation of evolution (6) and the nonlinear functionalequation (7) is basedupon the relations between the classes of nonexpansive and accretive mappings.

Browder [7] applied the theory of accretive operators to obtain a substantial strengthening of the fixed point theory of nonexpansive mappings as well as of a more general class of pseudo-contractive mappings.

Interest in Accretive mapping stems mainly from their firm connection with evolution equations. Many physically significant problems can be modeled by initial-value problems of the form

$$u'(t) + Au(t) = 0, u(0) = u_0 \quad (8)$$

where A is an accretive operator in an appropriate Banach space[32]. Typical examples where such evolution equations occur can be found in the heat, wave or Schrodinger equations[15, 19].

The solutions to the problem:

$$find\ u \in E\ such\ that\ Au = 0 \quad (9)$$

are precisely the equilibrium points of the system (8).

An early fundamental result in the theory of accretive operators, due to Browder [8], states that the initial value problem (8) has a solution if A is locally Lipschitzian and accretive on E. Utilizing the existence result for (8), Browder [6] proved that if A is locally Lipschitzian and accretive on E, then A is m-accretive. Martin [23] extended this result and proved that (8) is solvable if A is continuous and accretive on E and utilizing this result, he further proved that if A is continuous and accretive, then A is m-accretive which implies that the equation $x + Ax = f$ has a solution $x^* \in D(A)$, for any $f \in E$. This is one of the fundamental results of the theory of accretive operators.

In the evolution equation (8), if u is independent of t, then $\frac{du}{dt} = 0$ and the equation (8) reduces to (9) whose solution describes the equilibrium state or the stable state of the system described by (8). This is very important in many applications such as ecology, economics, and physics [32]. If A is accretive and $T = I - A$ is pseudocontractive then x^* is a solution of (9) if and only if x^* is a fixed point of T [7].

Browder [7] proved the following existence theorem for nonlinear accretive operators in Banach space.

Theorem 5.1 Let E be a Banach space, A a Lipschitzian mapping of E into E such that for all x and y in E,

$$(A(x) - A(y), w) \geq c_0||x - y||^2,$$

with a fixed constant $c_0 > 0$ where $w \in J(x - y)$. Then A maps E onto E. In other words "A Lipschitz strongly accretive operator has a zero in Banach space".

Other important results related to the existence of zeros of accretive operators and their variants include the results of [15, 17] and the references therein.

Deimling [15] gave some basic existence results about the zeros of strongly accretive operators:

Lemma 5.1 If E is a Banach space and $A : E \to E$ is continuous and strongly accretive, then $R(A) = E$. Hence, for any $f \in E$, the equation $Ax = f$ has at least one solution in E. Since A is strongly accretive, the solution must be unique.

Lemma 5.2 If E is uniformly smooth Banach space and $A: E \to E$ is strongly accretive and demi-continuous(i.e. $x_n \to x$ implies that $Ax_n \rightharpoonup Ax$), then A maps E onto E; that is, for each $f \in E$, the equation $Ax = f$ has a solution in E.

Using these results, the existence results for zeros of ϕstrongly accretive operators can also be obtained.

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