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Shear Deformation Mathematical Modeling of Functionally Graded Clamped Beams under bending

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ABSTRACT: In this paper, high order bending theories are used to develop an analytical model considering shear strains in displacement fields which have not been taken into account by other theories for functionally graded material (FGM) clamped symmetric beams under bending. A new polynomial shear function developed which represents the originality of this research work satisfies the boundary conditions and stress nullities on the lower and upper faces of the section through the thickness. These theories do not require a shear correction factor and consider a hyperbolic shape function. Material properties are assumed to vary in the thickness direction, a simple power-law distribution in terms of volume fractions of constituents is considered. An illustrative case is studied in this investigation, a clamped FGM beam subjected to a concentrated shear force at the middle, is presented the originality of this research work. The mathematical model is established by differential equations which are derived by the principle of virtual work. Equilibrium equations and boundary conditions are introduced. The solution model is based on a variation approach (integrals) to predict the field component of displacements and the basic constitutive laws. The solution of the analytical model is presented. The results in terms of displacement fields including rotation of the section, deformations, and stresses, predicted from the proposed model and compared to those of simply supported end beams found in the literature, are presented.

KEYWORDS: High order theories and bending, Mathematical model, Clamped Beams, New polynomial shear function, Displacement fields.

I. Introduction

Composite materials are widely used in industries because of their excellent properties of materials which result mainly from the best interaction between fibers and matrix and also the best properties of the constituents. In the research works of Shi (Shi et al., 2000) studied the shear deformations of beams with sixth-order differential equations at different boundary conditions the main objective is to present a new theory to analytically solve the sixth-order differential equilibrium equations of three typical shear deformable beams since the fourth-order theory of the Timoshenko beam theory (TBT) might generate some problems on the displacement boundary conditions. The solutions are more accurate than those given by the fourth-order differential equations of TBT, and agree well with the elasticity solutions. Lu (Lu et al., 2015) investigated the transverse shear deformation effects on the deflection of composite beams with various laminate configurations and boundary conditions. The main result obtained proves that the total deformation increases with increasing fiber orientation. The classical Euler-Bernoulli beam theory does not consider transverse shear, as opposed to TBT and higher order FGM beam theories which include shear strains and introduce a shape function. Many research works were carried out to develop mathematical models able to predict the static and dynamic response of thin and thick beams. Sankar (Sankar., 2000) proposed elasticity solution obtained for the simply supported functionally graded (FG) beams subjected to sinusoidal transverse load. The Euler–

Bernoulli beam type was also developed considering that the plane cross-sections remain plane after deformations, and the stresses and displacements depend on a non-dimensional parameter. Authors found that the FG beam theory is valid for long and slender beams with a slowly varying transverse load. Guenfoud (Guenfoud et al., 2016)proposed a new polynomial shear function which satisfies the stress-free boundary conditions. However, this theory has strong similarities with Timoshenko beam theory in some concepts such as equations of movement, boundary conditions and stress resultant expressions. Alshorbagy (Alshorbagy et al., 2007) studied free vibration characteristics and dynamic behavior of functionally graded beam using the finite element method for different boundary conditions. Significant conclusions were cited as the natural frequencies increase with an increase in power exponent (when $E_{ratio}< 1$), and decrease with an increase in power exponent (when $E_{ratio}> 1$).

Zhong (Zhong et al., 2007) proposed elasticity analytical solutions for functionally graded cantilever beams under different load types. Gao et al. (2007) were focused on the boundary conditions in the beam bending problem using the reciprocity theorem and the Papkovich-Neuber's solution to obtain the appropriate stress and the accurate mixed boundary conditions. A set of necessary conditions on the edge-data for the existence of a rapidly decaying solution, was established generalizing the method proposed by Gregory and Wan .Research works of Thai et al., Wei et al., Zaoui et al., and Ziou et al.(2012; 2017) carried out on the bending and free vibration of functionally graded material (FGM) beams. Theories of higher-order were used taking into account the transverse shear strain through the depth of the beam. The Analytical solutions were obtained for a simply supported beam. Authors concluded that an increase in the power law index results a decrease in the stiffness of FGM beam, and leads to an increase in the deflections and a reduction of the natural frequencies. The presence of cracks seems to reduce the frequencies and changes the vibration mode shapes of FGM beams, the shear deformation should be considered in the case that the slenderness ratio is less than some range ($L/h\leq10$). The deflections of short beams are higher than those of slender beams. The shear deformation effects are more evident for higher-mode frequencies than for lower-mode frequencies. Benatta (Benatta et al., 2021; 2018) studied the response of the bending of short hybrid composite beams varying fibers spacing. The authors proved that the hybrid FGM with a fiber volume fraction variation, can improve the beam design. Also, they carried out a static analysis of short FGM beams with simply supported ends including warping and shear deformations effects. The results obtained showed that the proposed formulation permits, for warping of the cross-section of the FGM beam, to eliminate the necessity to use the arbitrary shear correction coefficients. Mahi (Mahi et al., 2021) investigated the free vibration of FGM beams subjected to thermal stresses with general boundary conditions. The results showed the influences of temperature, constituent material distribution, and beam aspect ratio on the natural frequencies of the beam. Ghugal (Ghugal et al., 2021) proposed elasticity analytical solutions for the static flexure analysis of thick isotropic beams subjected to concentrate load and/or uniform distribution load with various end conditions and compared the displacement field of illustrative cases to those of Timoshenko and solutions in literature.

However, these previous research works as seen in this literature review did not investigate sufficiently the FGM beams with clamped ends taken into account the shear deformations in the displacement fields. Then, the objective of this paper is to propose an analytical model takinginto account the shear strain in the field of displacements for FGM clamped symmetric beams under three-point bending, such as illustrative case. Briefly, the manuscript is organized by description of material properties as well as the proposed mathematical model based on high-order bending theories to determine the governing differential equations which are derived by the virtual work principle and solved by integrals. Then, a numerical comparison and discussion of results, in terms of displacement fields, stresses, and deformations, predicted from the proposed model for beams with clamped ends and those available in the literature, are presented. This proposed analytical model using a new polynomial shear function is a new contribution.

II. Material and cross section properties

2.1 Material properties

The stiffness coefficients E(z) and G(z) obtained based on the mixing rule of constituents, can be written as follows:

$$E(z) = E_F V_F(z) + E_M(1 - V_F(z)) \quad (1)$$

$$\frac{1}{G(z)} = \frac{\boldsymbol{V_F(z)}}{G_F} + \frac{\mathbf{1} - V_F(z)}{G_M} \quad (2)$$

Where V_F: Fiber volume fraction; E_F: Young modulus of fiber; E_M: Young modulus of matrix; G_F: Fiber shear modulus; G_M: Matrix shear modulus.

Equation (3) which presents a simple power law with index *n* on the z-axis and the height of the section, following the constituent mixing rule, can be written as:

$$V_F(z) = V_2 + (V_1 - V_2)(2\frac{z}{h})^n \quad (3)$$

The fiber volume fraction V_F varies through the thickness, with values V_1=V_F(h/2) and V_2=V_F (0).

2.2Cross section properties

Equation (4) illustrates the stiffness coefficients of beam A_{11}, B_{11} and D_{11} which are extension, bending-extension coupling, and bending stiffness coefficients, respectively; its expression under integrals between the half–heights of the section is given by:

$$\{A_{11}, B_{11}, D_{11}\} = b\int_{z=\frac{-h}{2}}^{z=\frac{h}{2}} E(z)\{1, z, z^2\}dz \quad (4)$$

Although, B^a_{11}, D^a_{11}, F^a_{11} which represent the additional coupling coefficients and the bending stiffness depending on the shape function φ(z), are presented under the following integrals:

$$\{B^a{}_{11}, D^a{}_{11}, F^a{}_{11}\} = b\int_{z=\frac{-h}{2}}^{z=\frac{h}{2}} E(z)\,\varphi(z)\{1, z, \varphi(z)\}\,dz \quad (5)$$

Also, the coefficient of the additional transverse shear stiffness is:

$$\varphi(z) = z\left(\frac{h^2}{12} - \frac{13}{9}\right) - z^3\left(\frac{1}{9} - \frac{52}{27h^2}\right) \quad (14)$$

The new polynomial shear function proposed is

$$A^a{}_{55} = b\int_{z=\frac{-h}{2}}^{z=\frac{h}{2}} G(z)\left(\frac{\partial\varphi(z)}{\partial z}\right)^2 dz \quad (6)$$

III. Mathematical Model

3.1 Displacements fields

The displacement components are defined from the equations (7) to (11) along the x and z-directions of the beam, when the rotation of the section θ(x) is measured on the mean line:

$$u(x,z) = u_0(x) - z\frac{\partial w(x)}{\partial x} + \varphi(z)\left[\frac{\partial w(x)}{\partial x} + \theta(x)\right] \quad (7)$$

$$\mathrm{V}(x,z) = 0 \quad (8)$$

$$w(x,z) = w_0(x) \quad (9)$$

Deformations and stresses equations (10) to (13) which present the Green Lagrange linear strain tensor are cited as follows:

$$\varepsilon_x(x,z) = \frac{\partial u^0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \varphi(z)\frac{\partial \gamma_{xz}(x)}{\partial x} \quad (10)$$

$$\gamma_{xz}(x) = \frac{\partial\varphi(z)}{\partial z}\gamma^0{}_{xz}(x) \quad (11)$$

While, the normal stress and the shear stress of Hooke's basic constitutive law are presented as follows:

$$\sigma_{xx}(x) = E(z)\varepsilon_x \quad (12)$$

$$\tau_{xz}(x) = G(z)\gamma_{xz} \quad (13)$$

3.2New polynomial shear function

A new polynomial shear function is adopted in the present study cited as below:

compared to the parabolic shear deformation beam theories (PSDBT)suggested by Reddy [17]and cited with other functions in reference [1].The higher-order theories also including cross-sectional warping (shear deformation effect) added to displacement fields with nonlinear shape functions cited in equation (14) and Reddy equation (12-a), respectively.

$$\varphi(z) = z(1 - \frac{4z^2}{3h^2}) \qquad (14-a)$$

From equation 14, the new polynomial shear function as derived is given in equation (14-b) as follows:

$$\frac{\partial\varphi(z)}{\partial z} = \left(\frac{h^2}{12} - \frac{13}{9}\right) - z^2(\frac{1}{3} - \frac{52}{9h^2}) \qquad (14-b)$$

At both, top and bottom fiber when z=±h/2, for the first derived of polynomial shear function, the nullity of the shear stress at theses fibers can be confirmed. The equation (14-c) is the second derived for z=0 in which the shear stress is maximal at the midline fiber, when φ (z=0) =0.

$$\frac{\partial^2\varphi(z)}{\partial z^2} = -z(\frac{2}{3} - \frac{104}{9h^2}) \qquad (14-c)$$

Figure 1 shows the comparison between the new shear function and the Reddy's shear function. The Reddy' PSDBT function is suggested for real solution (Δ= 0). Also, it can be seen that the both shear functions are in good agreement for z=±h/2.

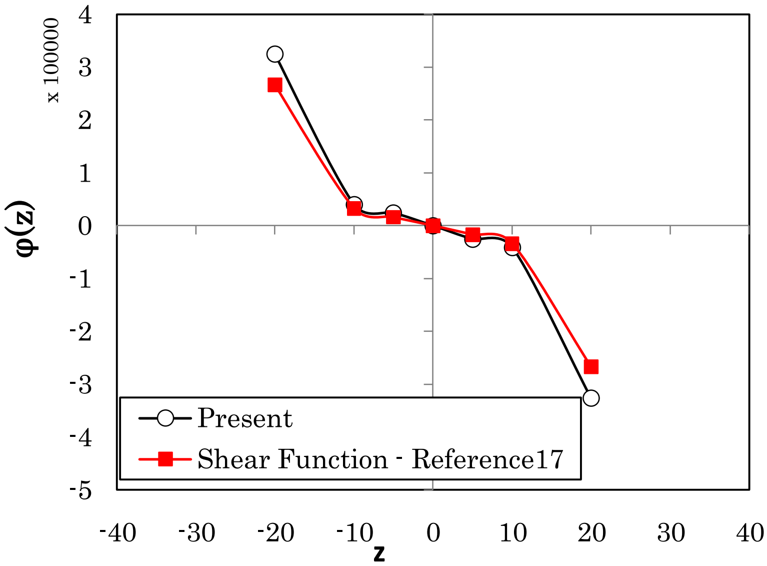

Figure 1 . Comparison between the new shear function and Reddy's shear function (Ref17)

IV. Displacements and Strains

4.1 Equilibrium equations

Consider the geometric dimensions of the beam at the boundary restrains, as shown in figure 2, having a width (b), span length (L), thickness (h), under force (F_Z). Figure 2 presents the applied loads and the symmetry section of the beam.

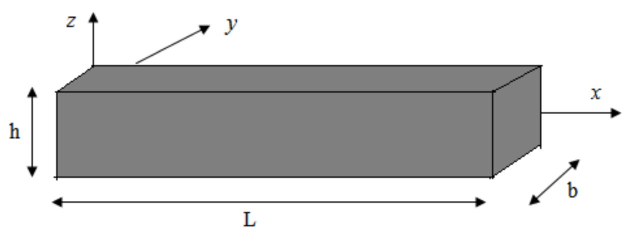

Figure 2. Geometry of beams

The virtual work principle applied in brief on the beam is defined as follows:

$$\boldsymbol{b}\int_{x=0}^{x=L}\int_{z=\frac{-h}{2}}^{z=\frac{h}{2}}(\sigma_x\delta\varepsilon_x+\tau_{xz}\delta\gamma_{xz})\boldsymbol{dxdz}-\int_{x=0}^{x=L}\boldsymbol{q}\delta\boldsymbol{w}\boldsymbol{dx}=\boldsymbol{0} \qquad \textbf{(15)}$$

$$b\int_{x=0}^{x=L}(N\frac{\partial\delta u_0}{\partial x}-M_b\frac{\partial^2\delta w_0}{\partial x^2}+M_s\frac{\partial\delta\psi_x}{\partial x}+Q\delta\psi_x)dx-b\int_{x=0}^{x=L}q\delta w_0dx=0 \qquad (15-a)$$

Equations (15-b) and (15-c) present the normal resultant force (N) and the shear resultant force (Q), respectively.

$$N=\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_x dz$$
(15-b)

$$Q=\int_{\frac{-h}{2}}^{\frac{h}{2}}\frac{\partial\varphi(z)}{\partial z}\tau_{zx}dz$$
(15-c)

The bending moment (M_b) and the shear bending moment (M_s), equations (15-d) and (15-e), respectively, are given by:

$$M_b=\int_{\frac{-h}{2}}^{\frac{h}{2}}z\sigma_x dz$$
(15-d)

$$M_s=\int_{\frac{-h}{2}}^{\frac{h}{2}}\varphi(z)\sigma_x dz$$
(15-e)

Equilibrium equations (16) to (19) are derived by deductions from the integrals:

$$\frac{\partial N(x)}{\partial x}=0$$
(16)

$$\frac{\partial^2(M(x)+M_1(x))}{\partial x^2}-\frac{\partial N(x)}{\partial x}+q=0$$
(17)

$$\frac{\partial(M(x)+M_1(x))}{\partial x}-N_1(x)=0$$
(18)

For virtual displacement equal to zero (δw (0) =0), where the concentrate load is applied:

$$\frac{\partial(M(x)+M_1(x))}{\partial x}-N_1(x)+F_z=0$$
(19)

The differential equations (20) to (23) are obtained under the following forms:

$$\frac{\partial}{\partial x}[(D_{11}-D^a{}_{11})\frac{\partial^3 w(x)}{\partial x^3}-(D^a{}_{11}-F^a{}_{11})\frac{\partial^2\gamma^0{}_{xz}(x)}{\partial x^2}=0 \qquad (20)$$

$$-D^a{}_{11}\frac{\partial^3 w(x)}{\partial x^3}-F^a{}_{11}\frac{\partial^2\gamma^0{}_{xz}(x)}{\partial x^2}=A^a{}_{55}\gamma^0{}_{xz}$$
(21)

With the integration of equation (20), a linear equation system is obtained:

$$D_{11}\frac{\partial^3 w(x)}{\partial x^3}+D^a{}_{11}\frac{\partial^2\gamma^0{}_{xz}(x)}{\partial x^2}=r_1$$
(22)

$$-D^a{}_{11}\frac{\partial^3 w(x)}{\partial x^3}-F^a{}_{11}\frac{\partial^2\gamma^0{}_{xz}(x)}{\partial x^2}=A^a{}_{55}\gamma^0{}_{xz}$$
(23)

4.2 Boundary conditions

Application of essential boundary conditions (Neumann) and natural (Dirichlet) for displacement and forces with a symmetry consideration:

- Case: Boundary conditions of clamped beam with concentrated load (P) at the middle

$$\frac{\partial w(x=L)}{\partial x}=w(x=0)=\theta(x=0)=0$$
(24)

4.3 Illustrative case

Equations (25), (26), (27) present displacements and deformations. Along the z-axis and the mean line of the beam for the interval (0≤ x ≤ L/2), we can write the following equation:

$$\upsilon(x)=-\upsilon(x)=x(4x^2-3Lx)$$
(25)

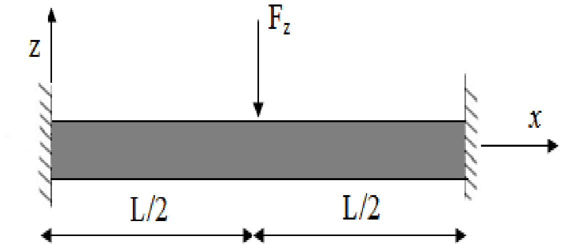

Figure 3. Clamped beam with concentrate load F_z at the middle

When the shear phenomenon is taken into account, the dimensionless coefficient is as follows:

$$S_\theta = \frac{48}{L^2}\frac{{D^a}_{11}{}^2}{{A^a}_{55}D_{11}} \quad (26)$$

The function φ_0 (z) which is derived from the solution can be defined by the following expression:

$$\varphi_\theta(\mathrm{x}) = x + \frac{1}{\Omega_\theta}(\cosh(\Omega_\theta x) - \sinh(\Omega_\theta x) - 1) \quad (27)$$

Finally, the displacement components are as follows:

$$u(\mathrm{x},\mathrm{z}) = -\mathrm{z}\frac{\mathrm{F_z}L^2}{192bD_{11}}\left(\frac{\partial \upsilon(\mathrm{x})}{\partial x} + 2S_\theta\frac{\partial\varphi_\theta(\mathrm{x})}{\partial x}\right) + \varphi(\mathrm{z})\frac{\mathrm{F_z}}{2\mathrm{b}{D^a}_{11}}\left(\frac{L^2}{48}S_\theta\frac{\partial\varphi_\theta(\mathrm{x})}{\partial x}\right) \quad (28)$$

$$V(\mathrm{x},\mathrm{z}) = 0 \quad (29)$$

$$w(\mathrm{x}) = \frac{\mathrm{F_z}L^2}{192bD_{11}}[\upsilon(\mathrm{x}) + 2S_\theta\varphi_\theta(\mathrm{x})] \quad (30)$$

However, the deformation components along x and z-directions of the beam are given as follows:

$$\varepsilon(\mathrm{x},\mathrm{z}) = -\mathrm{z}\frac{\mathrm{F_z}L^2}{192bD_{11}}\left(\frac{\partial^2\upsilon(\mathrm{x})}{\partial x^2} + 2S_\theta\frac{\partial^2\varphi_\theta(\mathrm{x})}{\partial x^2}\right) + \varphi(\mathrm{z})\frac{\mathrm{F_z}}{2\mathrm{b}{D^a}_{11}}\left(\frac{L^2}{48}S_\theta\frac{\partial^2\varphi_\theta(\mathrm{x})}{\partial x^2}\right) \quad (31)$$

Furthermore, the transverse shear deformation is written as:

$$\gamma^0{}_{\mathrm{xz}}(x,z) = \frac{\partial\varphi(z)}{\partial z}\frac{\mathrm{F_z}L^2}{96\mathrm{b}{D^a}_{11}}S_\theta\frac{\partial\varphi_\theta(\mathrm{x})}{\partial x} \quad (32)$$

Let us defined the newly proposed shear coefficient S:

$$\mathrm{S} = \frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2 = \frac{\mathrm{bh}}{\mathrm{I}} * \frac{{F^*}_{55}}{{D^*}_{11}}\left(\frac{h}{L}\right)^2 \quad (33)$$

Where, ${F^*}_{55} = \frac{1}{hG_{xz}}$; ${D^*}_{11} = \frac{\mathrm{b}}{E_x I}$ are the shear stiffness constant and the bending stiffness constant of the beam, respectively.

The transverse expression of the displacement (30) can re-write (see appendix-2) and present by the following equation:

$$\mathrm{w_0(x)} = \frac{\mathrm{P}L^2x}{16E_xI}[1 - \frac{1}{6}\frac{x^2}{L^2} + \frac{16}{24}\mathrm{S}] \quad (34)$$

At(x=L/2) for (eq.34), the maximum value is evaluated as follows:

$$\mathrm{w}_C\left(\mathrm{x} = \frac{\mathrm{L}}{2}\right) = \frac{\mathrm{P}L^3}{192E_xI}[\frac{23}{24} + 4\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2] \quad (34\text{-a})$$

V. Discussion of Numerical Results

The results obtained in this study are compared to those of simply supported beams cited in the literature [Benatta et al-2008] for the FGM beam using the following numerical characteristics: L=20mm; b=30mm; h=4mm; F_Z =5kn; E_f=138GPa; E_m=3.5GPa; G_F=12GPa; G_m=1.6GPa; and fraction volume with V_1=0.6 and V_2=0.4.

In figure 4, the comparison of the results illustrated in terms of displacements predicted from the proposed analytical model and those obtained from the literature model, can show that there is a significant decrease by a quarter, when the beam boundary conditions are changed from the simply supported ends to clamped edges through the z-axis.

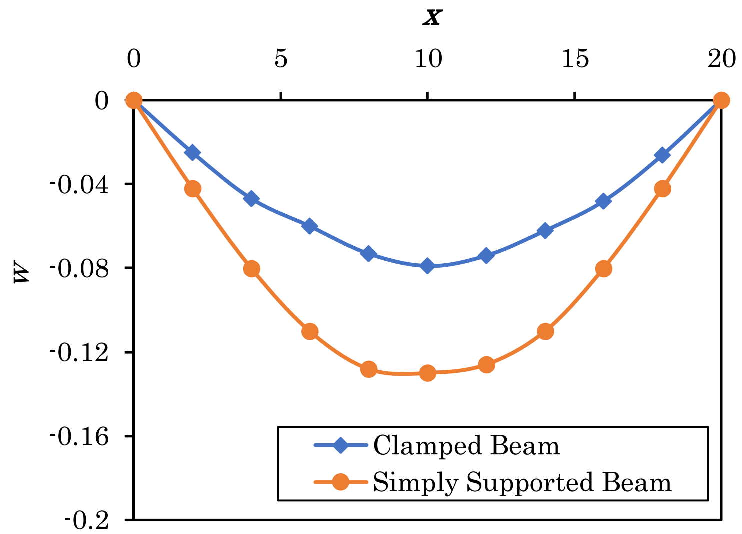

Figure 4. Comparisonofz-axis displacement(w) betweenclamped beam and simply supported beam

From figure 5, it can be observed that the x-axis displacement curves through the thickness exhibit a cubic distribution for higher-order theories (PDSBT) used. However, for classical Euler-Bernoulli and Timoshenko theories used, the displacement curves show a linear distribution.

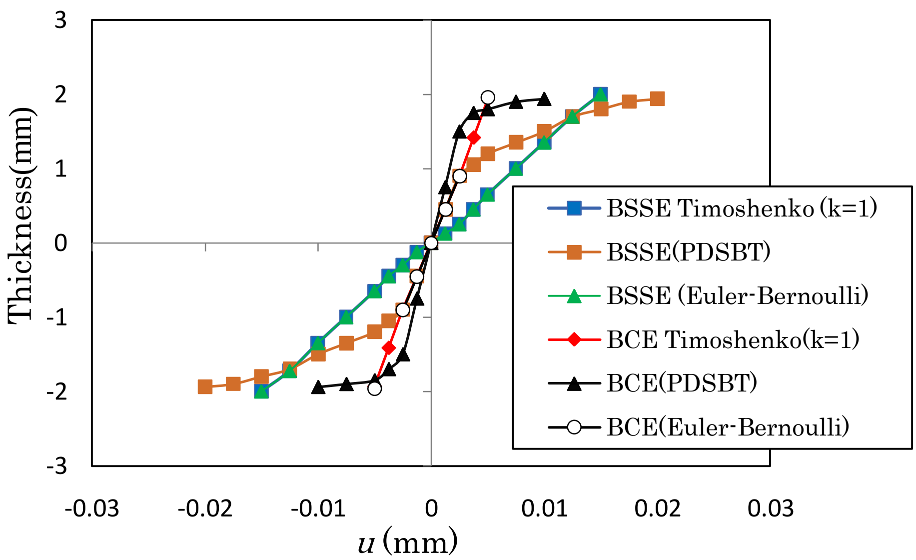

Figure 5. Comparison of longitudinal displacement (u) betweenclamped beam ends and simply supported beams theories

In figure 6, the comparison shows a linear stress increase for Timoshenko's theory used with a correction coefficient (k=5/6). Nevertheless, a cubic distribution of stress is observed for higher-order theories (PDSBT) used.

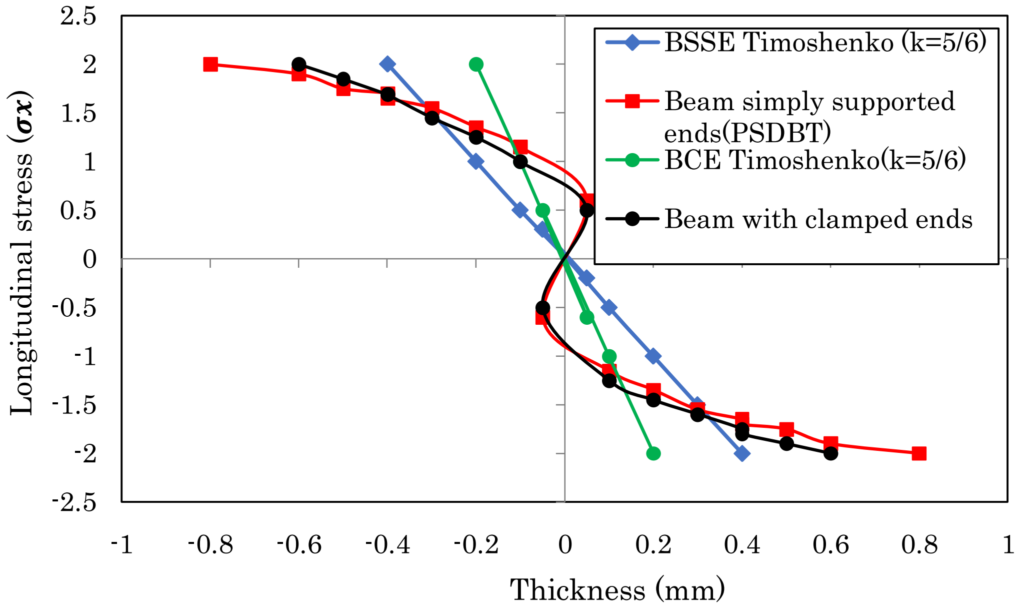

Figure 6 . Comparison of variation longitudinal stress (σ_x)betweenclamped ends beam and simply supported beams theories

In figure 7, the comparison shows a cubic distribution of the transverse shear stresses using higher-order theories (PDSBT). However, the shear stress distribution through the beam thickness is constant using the Timoshenko theory.

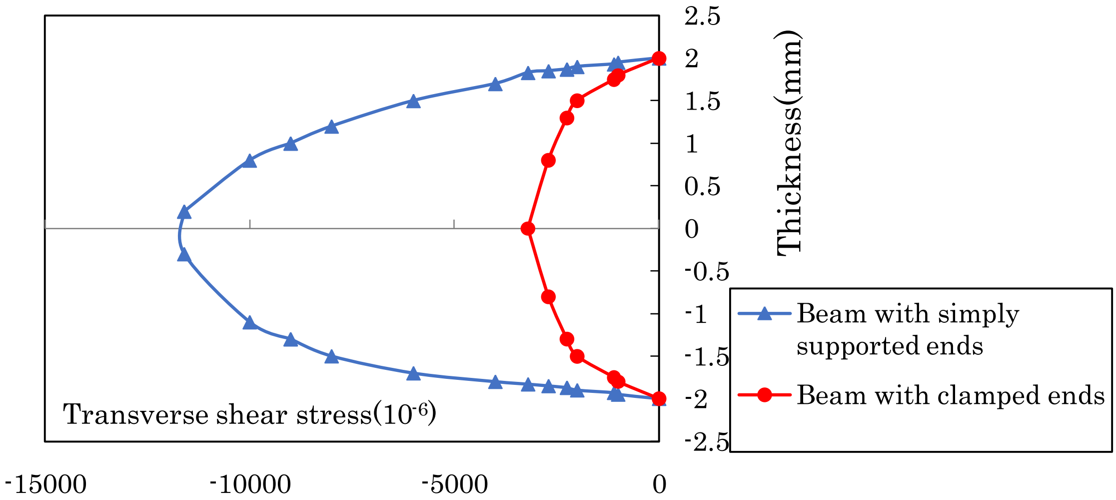

Figure 7 . Comparison of transverse shear stress (τ_{xz}) between clamped beam ends and simply supported beams theories

Considering illustrative example of the clamped steel beam subjected to concentrate load F_z (figure3), from Equation (35) proposed by Ghugal and Sharma (2011), the maximum transverse displacement for the flexure of thick beams is given by the following expression:

$$w_{c2}\left(x = \frac{L}{2}\right) = \frac{PL^3}{192E_xI}\left[1 + 9.6(1+\mu)\left(\frac{h}{L}\right)^2\right]$$
(35)

Where, µ is the Poisson’s ratio of the beam material.

The properties of the steel beam (IPE 330) considered are as follows:
Cross-section bxh(160x330 mm^2); Length L = 10 m; Load F_z= 5 kN(including the self-weight of the beam and the pedestrian load); Moment of inertia I=11770 cm^4; Poisson's ratio μ=0.3; Elasticity modulus E_x=210GPa, for type of steel beam: S275. The displacement value at mid-
Span obtained from equation (35) (w_{c2}=0,1067mm) is similar to that predicted from the proposed model (Eq. 34-a) (w_{c2}=0,1021mm). It can be observed that the difference between both values less than 5%.This implies the validity of the proposed model.

VI. Conclusions

A new analytical model is proposed taking into account shear strains in displacement fields for FGM beams with clamped ends under bending. The results in terms of displacement fields including shear strains and section rotation, strains and stresses, are presented. Higher-order shear strains new theories are used which include shape functions and stress nullities at the bottom and top faces of the section through the beam thickness. These theories do not require a shear correction coefficient contrary to the Timoshenko's beam theory, and also to Euler-Bernoulli classical beam theories (CBT). A fruitful numerical comparison is carried out to compare results predicted from the proposed model for clamped ends beam with those obtained from the literature for simply supported beams, in terms of displacements, deformations, axial and shear stresses. The present mathematical model could be used by designers in composite materials field to more understand the flexural behavior of FGM beams. In fact, the results obtained in this study allow draw the following conclusions:

(1) The proposed high order model taking into account the shear strain in displacement fields for flexural FGM clamped ends beams exhibits more precise results (nonlinear behavior) with respect to those predicted from Timoshenko beam theory (linear behavior) in which shear strain effect is considered by a correction coefficient and also Euler-Bernoulli classical beam theory (CBT)which neglects shear strains in displacement fields.
(2) The results in terms of displacements predicted from the proposed analytical model and the literature model, show that there is a remarkable decrease by a quarter when the boundary conditions of the beam is changed from simply supported ends to clamped ends.
(3) The proposed mathematical model considering the shear strain in displacement fields is efficient to predict the solution for FGM beams with clamped ends under three-point bending.
(4) The new polynomial shear function exhibits a good convergence with those of the other higher order shear deformation beam theories.

For future works, it is recommended to analyze the proposed model for other materials of beams and shells not used in this study.

Appendix-1: Case

$N_1(x)$ and $M_1(x)$ are the resultant force, and the resultant moment, respectively, depending on the additional shape function, and are defined by the following equations:

$$N_1(x) = \int \frac{\partial \varphi(z)}{\partial z} \tau_{xz}\, dS = b[A^*{}_{55}\gamma^0{}_{xz}(x)]$$

$$M_1(x) = \int \varphi(z)\sigma_x\, dS = b[B^a{}_{11}\frac{\partial u^0(x)}{\partial x} - D^a{}_{11}\frac{\partial^2 w}{\partial x^2} + F^a{}_{11}\frac{\partial \gamma^0{}_{xz}(x)}{\partial x}]$$

$$\frac{\partial \varphi_\theta(x)}{\partial x} = \sinh \Omega_\theta x - \cosh \Omega_\theta x - 1$$

$$u(x,z) = -z\frac{F_z}{2D_{11}}[-\frac{x^2}{4} + \frac{L}{8}x + \frac{48}{L^2}\frac{D^a{}_{11}{}^2}{A^a{}_{55}D_{11}}(1 - \cosh(\Omega_\theta x) + \sinh(\Omega_\theta x) - \frac{\sinh(\Omega_\theta x)}{\sinh(\Omega_\theta \frac{L}{2})}] + \varphi(z)\frac{F_z}{2b}\frac{D^a{}_{11}}{A^a{}_{55}D_{11}}(1 - \cosh(\Omega_\theta x) + \sinh(\Omega_\theta x) - \frac{\sinh(\Omega_\theta x)}{\sinh(\Omega_\theta \frac{L}{2})})$$

$$w(x) = \frac{F_z}{D_{11}}[\ -\frac{1}{12}x^3 + \frac{L}{16}x^2 + \frac{48}{L^2}\frac{D^a{}_{11}{}^2}{A^a{}_{55}D_{11}}[x + \frac{1}{\Omega_\theta}(\cosh(\Omega_\theta x) - \sinh(\Omega_\theta x) - 1)]\ ]$$

$$\theta(x) = \frac{F_z}{2}\frac{D^a{}_{11}}{A^a{}_{55}D_{11}}\left[-\frac{A^a{}_{55}}{4D^a{}_{11}}x^2 + \frac{LA^a{}_{55}}{8D^a{}_{11}}x + \frac{48}{L^2}\frac{D^a{}_{11}}{D_{11}}[1-\cosh(\Omega_\theta x) + \sinh(\Omega_\theta x)]\right]$$

Appendix-2: Case

It can be noted that only a pure flexure is reported.

For cut 0≤ x≤ L/2, the moment of bending is as the following:

$$M(x) = \frac{P}{2}x - \frac{P}{8}L \quad (1)$$

Integrate the previous equation and deduce the expression for φ_x:

$$E_x I\varphi_x = \frac{P}{4}x^2 - \frac{PL}{8}x + c_1 \quad (2)$$

The symmetry of the deformation requires:

$$\varphi_x\left(\frac{L}{2}\right) = 0 \quad (3)$$

Equation (2) can be written:

$$E_x I\varphi_x = \frac{P}{4}x^2 - \frac{P}{8}L\,x \quad (4); \quad \varphi_x = \frac{PL^2}{16E_x I}\left(1 - \frac{1}{2}\frac{x^2}{L^2}\right) \quad (4.a)$$

The first derivation of the displacement expression can be taken as the following:

$$\frac{\partial w_0}{\partial x} = -\left(\varphi_x + \frac{P}{2bhG_{xz}}\right) \quad (5) \quad \Rightarrow \frac{\partial w_0}{\partial x}\left(\frac{L}{2}\right) = -\frac{P}{2bhG_{xz}} \quad (5\text{-a})$$

The final integrate expression is:

$$w_0(x) = -\int_0^{\frac{L}{2}} \frac{PL^2}{16E_x I}\left(1 - \frac{1}{2}\frac{x^2}{L^2} + \frac{P}{2bhG_{xz}}\right)dx \quad (6)$$

The final displacement is reported in the expression (eq.34)

The final maximum displacement is reported into (eq.34-a) at the mid-span.

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